Analiza matematyczna 1
Zajęcia nr 4

Granice

```
? Limit
```

Limit[$expr$, $x \to x_0$] finds the limiting value of $expr$ when x approaches x_0. >>

```
Limit[Sin[x] / x, x → Infinity]
```

0

Założenie:

0

```
Limit[x^a, x → Infinity, Assumptions → a == 0]
```

1

```
Limit[x^a, x → Infinity, Assumptions → a > 0]
```

∞

Granice prawostronne i lewostronne

lewostronne:

```
Limit[1 / x, x → 0, Direction → 1]
```

$-\infty$

prawostronne:

```
Limit[1 / x, x → 0, Direction → -1]
```

∞

Zadanie

policzyc granice

a) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

b) $\lim_{n\to\infty} \frac{\sin(n!)}{\sqrt{n}}$

c) $\lim_{n\to\infty} \sqrt{1+n} - \sqrt{n}$

d) $\lim_{n\to\infty} \frac{1}{n^q}$ dla q>0

e) $\lim_{n\to\infty} \frac{(2n+1)!-(2n-1)!}{(2n)!\,n!}$

f) $\lim_{n\to\infty} \frac{c^n}{n^k}$ dla c>1

g) $\lim_{x \to \infty} \sin(x)$

h) $\lim_{x \to 0} \frac{\sin(x)}{x}$

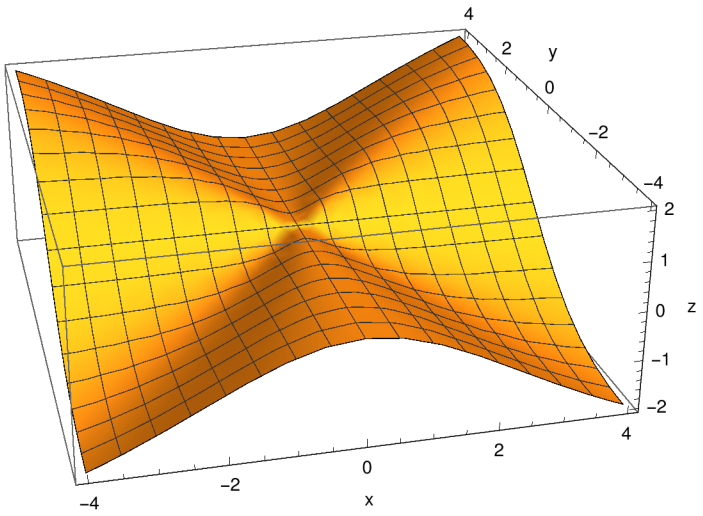
Granicefunkcjidwóchzmiennych

$\lim_{x \to x0, y \to y0} f(x, y)$

```
A[x_, y_] = x^2 y / (x^2 + y^2)
```

$$\frac{x^2 y}{x^2 + y^2}$$

```
Plot3D[A[x, y], {x, -4, 4}, {y, -4, 4},
 AxesLabel → {"x", "y", "z"}]
```



Przejście do zmiennych biegunowych

```
AA[r_, θ_] = A[x, y] /. {x → r Cos[θ], y -> r Sin[θ]}
```

$$\frac{r^3 \operatorname{Cos}[\theta]^2 \operatorname{Sin}[\theta]}{r^2 \operatorname{Cos}[\theta]^2 + r^2 \operatorname{Sin}[\theta]^2}$$

```
AA[r_, θ_] = Simplify[%]
```

$$r \operatorname{Cos}[\theta]^2 \operatorname{Sin}[\theta]$$

```
? Simplify
```

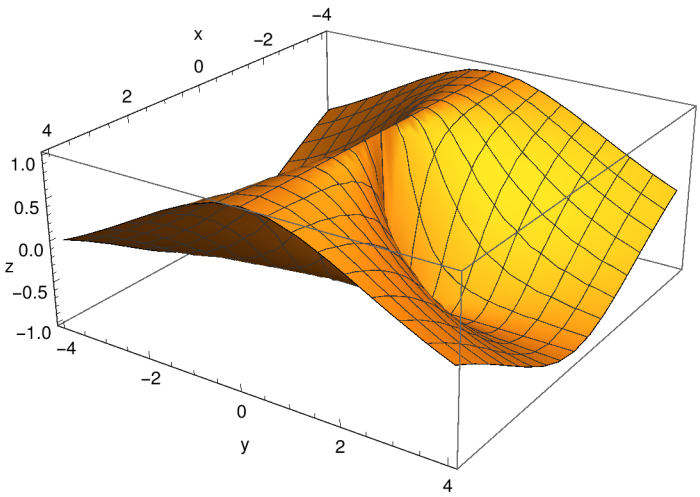
Simplify[*expr*] performs a sequence of algebraic and other transformations on *expr* and returns the simplest form it finds.

Simplify[*expr*, *assum*] does simplification using assumptions. >>

```
B[x_, y_] = (x^2 - y^2) / (x^2 + y^2)
```

$$\frac{x^2 - y^2}{x^2 + y^2}$$

```
Plot3D[B[x, y], {x, -4, 4}, {y, -4, 4},
 AxesLabel → {"x", "y", "z"}]
```




```
BB[r_, θ_] = Simplify[ B[x, y] /. {x → r Cos[θ], y -> r Sin[θ]}]
```

Cos[2 θ]

```
Limit[BB[r, 0] , r -> 0]
```

1

```
Limit[BB[r, Pi / 2] , r -> 0]
```

-1

Zadanie

1. Zdefiniować funkcję $G(x, y) = \frac{x\,y}{x^3+y^2}$

2. Narysować wykres 3D dla x∈[-4,4], y∈[-4,4]
3. Przechodząc do zmiennych biegunowych zbadać granicę w punkcie {0,0}

Sumawyrazowciągu

```
? Sum
```

Sum[f, {i, i_{max}}] evaluates the sum $\sum_{i=1}^{i_{max}} f$.

Sum[f, {i, i_{min}, i_{max}}] starts with $i = i_{min}$.
Sum[f, {i, i_{min}, i_{max}, di}] uses steps di.
Sum[f, {i, {i_1, i_2, …}}] uses successive values i_1, i_2, … .

Sum[f, {i, i_{min}, i_{max}}, {j, j_{min}, j_{max}}, …] evaluates the multiple sum $\sum_{i=i_{min}}^{i_{max}} \sum_{j=j_{min}}^{j_{max}} \ldots f$.

Sum[f, i] gives the indefinite sum $\sum_i f$. »

```
Sum[k^2, {k, 10}]
```

385

```
Sum[k^2, {k, n}]
```

$\frac{1}{6} n (1 + n) (1 + 2 n)$

```
Sum[q^k, {k, 0, n}]
```

$\frac{-1 + q^{1+n}}{-1 + q}$

```
Sum[1 / (2 k + 1)^2, {k, 0, Infinity}]
```

$\frac{\pi^2}{8}$

Zadanie

Obliczyć sumy:

a) $\sum_{k=1}^{\infty} \frac{(-1)^{1+k}}{k}$
b) $\sum_{k=0}^{\infty} \frac{1}{2^k}$
c) $\sum_{k=0}^{\infty} \frac{2^k - 3^k}{5^k}$
d) $\sum_{k=1}^{\infty} \frac{\mathrm{Log}[k]}{k^2}$

Pochodne

```
? D
```

D[f, x] gives the partial derivative $\partial f / \partial x$.
D[f, {x, n}] gives the multiple derivative $\partial^n f / \partial x^n$.
D[f, x, y, ...] differentiates f successively with respect to x, y, ...
D[f, {{x_1, x_2, ...}}] for a scalar f gives the vector derivative ($\partial f / \partial x_1$, $\partial f / \partial x_2$, ...).
D[f, {*array*}] gives a tensor derivative. >>

```
f[x_, y_] = 1 / (x^2 + y)
```

$$\frac{1}{x^2 + y}$$

```
D[f[x, y], x]
```

$$-\frac{2\,x}{(x^2 + y)^2}$$

```
D[f[x, y], {x, 2}]
```

$$\frac{8\,x^2}{(x^2 + y)^3} - \frac{2}{(x^2 + y)^2}$$

```
D[f[x, y], x, y]
```

$$\frac{4\,x}{(x^2 + y)^3}$$

```
D[f[x, y], {x, 2}, y]
```

$$-\frac{24\,x^2}{(x^2 + y)^4} + \frac{4}{(x^2 + y)^3}$$

Zadanie

Oblicz pochodną nastepujących funkcji

a) f (x) = $\frac{\mathrm{tg}\,(x) - \mathrm{ctg}\,(x)}{\mathrm{tg}\,(x) + \mathrm{ctg}(x)}$, oblicz $\frac{d}{dx} f$

b) f (x) = $\frac{\arcsin(x)}{x^2}$, oblicz $\frac{d^2}{dx^2} f$

c) f (x,y) = $\arcsin\left(\frac{y}{x^2}\right)$, oblicz $\frac{d^2}{dx\,dy} f$

Styczna i normalna do krzywej

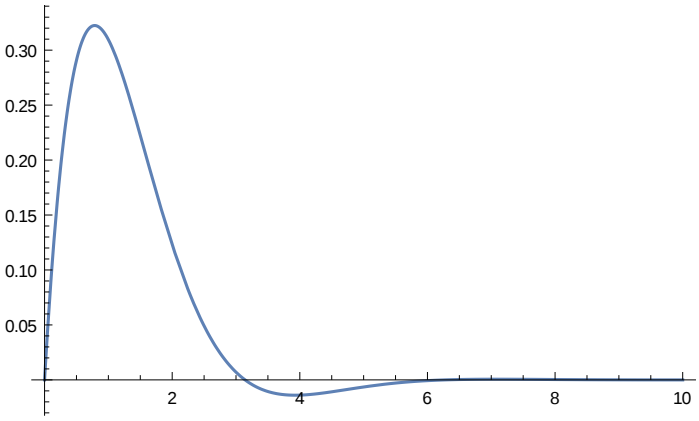
```
f[x_] = Exp[-x] Sin[x]
```

e^{-x} Sin[x]

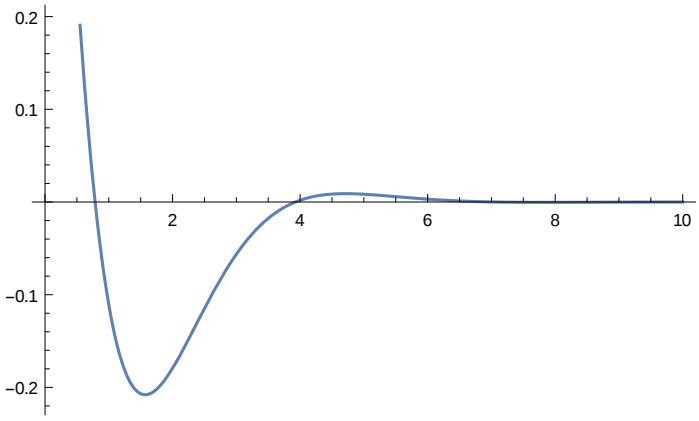
```
p[x_] = D[f[x], x]
```

e^{-x} Cos[x] - e^{-x} Sin[x]

```
Plot[f[x], {x, 0, 10}]
```




```
Plot[p[x], {x, 0, 10}]
```



Styczna do wykresu funkcji f w punkcie x_0

```
styczna[x_, x0_] = f[x0] + p[x0] * (x - x0)
```

$$e^{-x0}\,\mathrm{Sin}[x0] + (x - x0)\left(e^{-x0}\,\mathrm{Cos}[x0] - e^{-x0}\,\mathrm{Sin}[x0]\right)$$

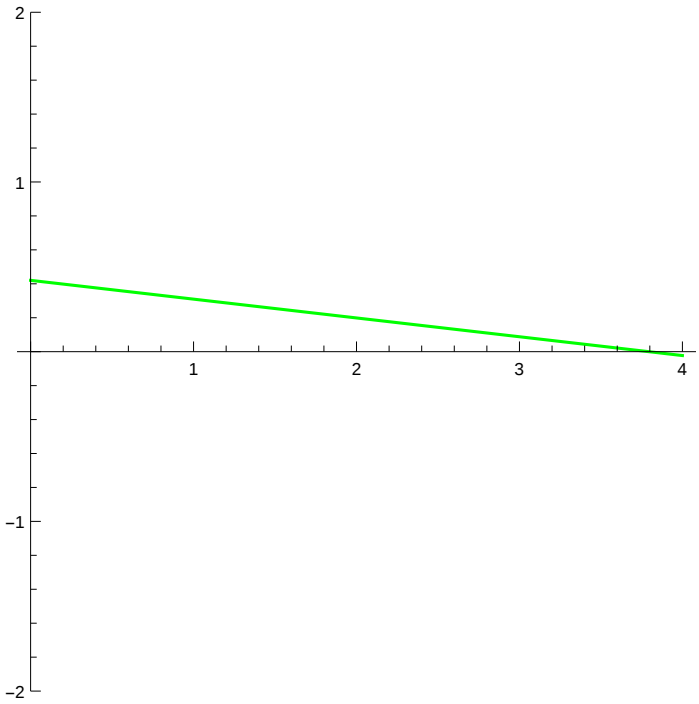
Normalna do wykresu funkcji f w punkcie x_0

```
normalna[x_, x0_] = f[x0] - (1 / p[x0]) * (x - x0)
```

$$e^{-x0}\,\mathrm{Sin}[x0] - \frac{x - x0}{e^{-x0}\,\mathrm{Cos}[x0] - e^{-x0}\,\mathrm{Sin}[x0]}$$

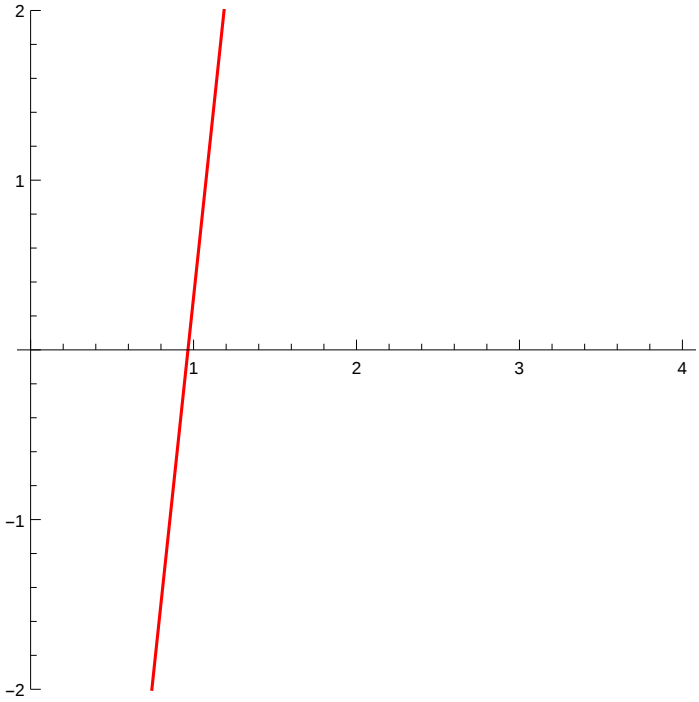
Wykres stycznej do wykresu funkcji f w punkcie $x = 1$,y=f(1)

```
Plot[styczna[x, 1], {x, 0, 4},
 PlotRange → {-2, 2}, AspectRatio → 1, PlotStyle → Green]
```

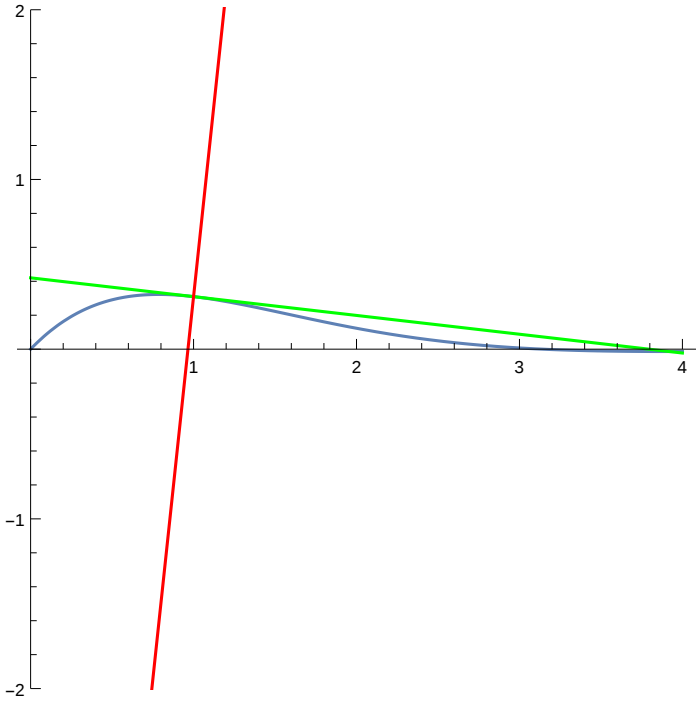


Wykres normalnej do wykresu funkcji f w punkcie $x = 1$,y=f(1)

```
Plot[normalna[x, 1], {x, 0, 4},
 PlotRange → {-2, 2}, AspectRatio → 1, PlotStyle → Red]
```




```
Show[
 Plot[f[x], {x, 0, 4}, PlotRange → {-2, 2}, AspectRatio → 1],
 Plot[styczna[x, 1], {x, 0, 4}, PlotStyle → Green],
 Plot[normalna[x, 1], {x, 0, 4}, PlotStyle → Red]
]
```



```
? Maximize
```

Maximize[f, x] maximizes f with respect to x.
Maximize[f, {x, y, …}] maximizes f with respect to x, y, ….
Maximize[{f, $cons$}, {x, y, …}] maximizes f subject to the constraints $cons$.
Maximize[…, $x \in reg$] constrains x to be in the region reg.
Maximize[…, …, dom] constrains variables to the domain dom, typically Reals or Integers. ≫

```
Maximize[{p[x], 0 ≤ x ≤ 10}, x]
```

{1, {x → 0}}

```
Minimize[{p[x], 0 ≤ x ≤ 10}, x]
```

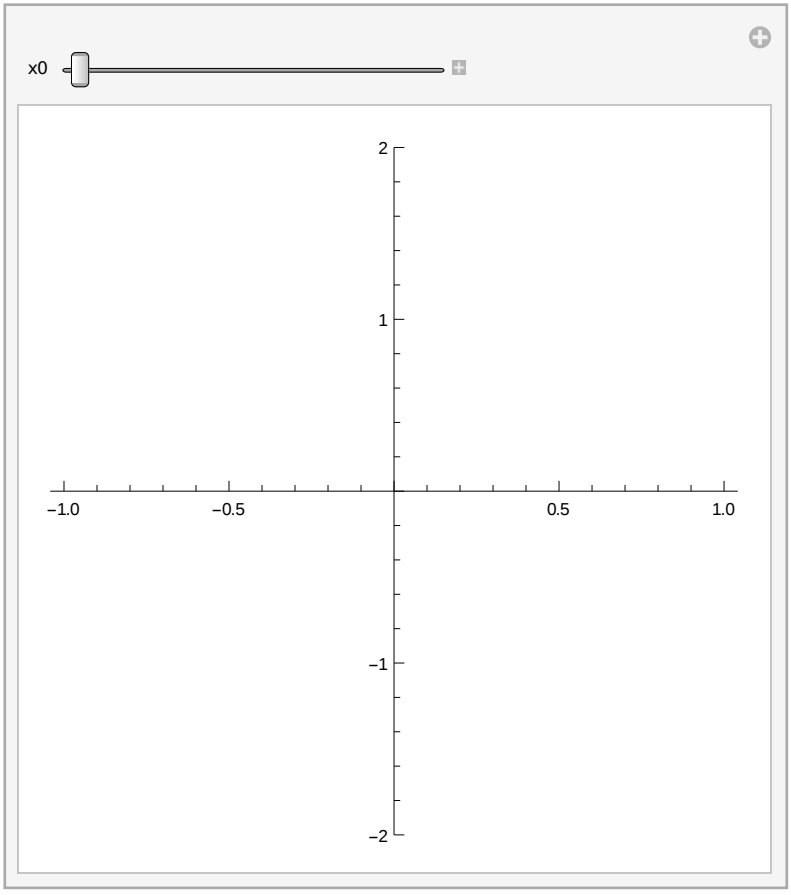
$$\left\{-e^{-\pi/2}, \left\{x \to \frac{\pi}{2}\right\}\right\}$$

Wykresz suwakiem

```
? Manipulate
```

Manipulate[$expr$, {u, u_{min}, u_{max}}] generates a version of
$expr$ with controls added to allow interactive manipulation of the value of u.
Manipulate[$expr$, {u, u_{min}, u_{max}, du}] allows the value of u to vary between u_{min} and u_{max} in steps du.
Manipulate[$expr$, {{u, u_{init}}, u_{min}, u_{max}, …}] takes the initial value of u to be u_{init}.
Manipulate[$expr$, {{u, u_{init}, u_{lbl}}, …}] labels the controls for u with u_{lbl}.
Manipulate[$expr$, {u, {u_1, u_2, …}}] allows u to take on discrete values u_1, u_2, ….
Manipulate[$expr$, {u, …}, {v, …}, …] provides controls to manipulate each of the u, v, ….
Manipulate[$expr$, c_u → {u, …}, c_v → {v, …}, …]
links the controls to the specified controllers on an external device. ≫

```
Manipulate[
 Show[
  Plot[f[x], {x, 0, 4}, PlotRange → {-2, 2}, AspectRatio → 1],
  Plot[styczna[x, x0], {x, 0, 4}, PlotStyle → Green],
  Plot[normalna[x, x0], {x, 0, 4}, PlotStyle → Red]
 ],
 {x0, 0, 4}]
```




```
? DynamicModule
```

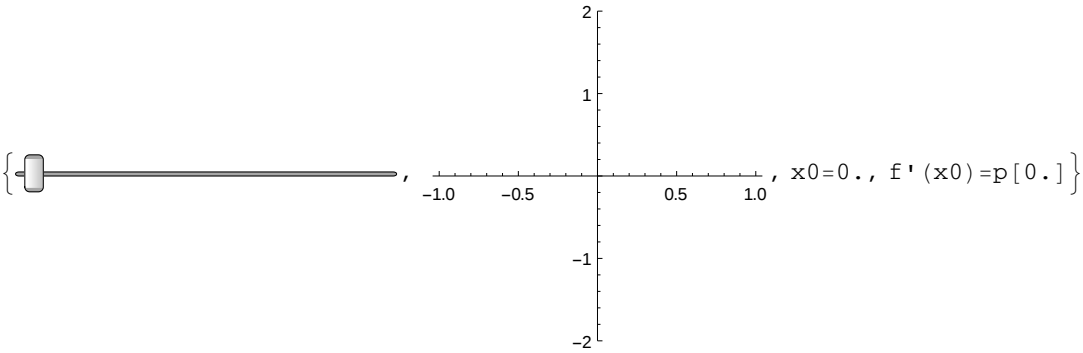
DynamicModule[{x, y, …}, $expr$] represents an object which maintains the same local instance of the symbols x, y, … in the course of all evaluations of Dynamic objects in $expr$. Symbols specified in a DynamicModule will by default have their values maintained even across Wolfram System sessions.

DynamicModule[{$x = x_0$, $y = y_0$, …}, $expr$] specifies initial values for x, y, …. ≫

```
DynamicModule[{x0},
 {

  Slider[Dynamic[x0], {0, 4}],
  Dynamic[
   Show[
    Plot[f[x], {x, 0, 4}, PlotRange → {-2, 2}, AspectRatio → 1],
    Plot[styczna[x, x0], {x, 0, 4}, PlotStyle → Green],
    Plot[normalna[x, x0], {x, 0, 4}, PlotStyle → Red]
   ]],
  Dynamic["x0=" <> ToString[x0]],
  Dynamic["f'(x0)=" <> ToString[p[x0]]]

 }
]
```



Zadanie

1. Zdefiniuj funkcję
f(x)=x- arctan(x)
oblicz pochodną f'(x)
oraz jej styczną i normalną do wykresu w punkcie x_0 (funkcja x i x_0).

2. Stwórz DynamicModule zawierający:
- wykres f, jej stycznej x∈ {-1,3} oraz normalnej
- suwak zmieniający wartość x_0 od -1 do 3
- wartość x_0 oraz pochodnej $f'(x_0)$

3. Znajdz najwieksza i najmniejsza wartosc pochodnej w przedziale{-1,3} (funkcje Maximize, Minimalize)