Symmetry Breaking in Quantum Systems

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Abstract

Dynamical symmetry in physical systems. Free massive scalar particles interacting with each other - Spontaneous Symmetry Breaking. The Linear Sigma Model - breaking continuous symmetry. Spontaneous continuous symmetry breaking of massless particles - Goldstone's theorem.

1 Symmetry in physics

The realization of symmetries in physical systems has proven to be of help in the description of physical phenomena: it makes it possible to relate the behaviour of similar systems and therefore it leads to a great simplification of the mathematical description of Nature.

The simplest concept of symmetry occurs at the geometrical or kinematical level when the shape of an object or the configuration of a physical system is invariant or symmetric under geometric transformations like rotations, reflections etc. At the dynamical level, a system is symmetric under a transformation of the coordinates or of the parameters which identify its configurations, if correspondingly its dynamical behaviour is symmetric in the sense that the action of the symmetry transformation and of time evolution commute.

To formalize the concept of dynamical symmetry, we first recall that the description of a classical physical system consists in

i) the identification of all its possible configurations S_γ, with γ running over an index set of coordinates or parameters which identify the configuration S_γ

ii) the determination of their time evolution

$$\alpha^t : S_\gamma \to \alpha^t S_\gamma \equiv S_{\gamma(t)}. \tag{1}$$

A symmetry g of a physical system is a transformation of the coordinates (or of the parameters) γ, $g : \gamma \to g\gamma$, which

1) induces an invertible mapping of configurations

$$g : S_\gamma \to gS_\gamma \equiv S_{g\gamma} \tag{2}$$

2) does not change the dynamical behaviour, namely

$$\alpha^t g S_\gamma = \alpha^t S_{g\gamma} \equiv S_{(g\gamma)(t)} = S_{g\gamma(t)} = g\alpha^t S_\gamma. \tag{3}$$

The above condition states that the symmetry transformation commutes with time evolution. For classical canonical systems, this amounts to the invariance of the Hamiltonian under the symmetry g (symmetric Hamiltonian).

The realization of a symmetry which relates (the configurations of) two seemingly different systems clearly leads to a unification of their description. In particular, the solution of the dynamical problem for one configuration automatically gives the solution for the symmetry related configuration (see (3)). [3, p. 7]

2 Spontaneous Symmetry Breaking

For free scalar particles of mass μ we have the Lagrangian:

$$L = T - V = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 \tag{4}$$

We are consider scalar particles that interact with each other. Then V in (4) must contain an extra term of the form ϕ^4. This Lagrangian has a discrete symmetry, it is invariant under the transformation $\phi \to -\phi$. Thus the most general Lagrangian for the scalar field would be:

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}(\phi^4) \tag{5}$$

where μ is the particle mass. L has dimensions of energy per unit volume, or E^4, while the boson field ϕ clearly has dimensions of E. Thus λ is a dimensionless constant.

The minimum value of V occurs at $\phi = \phi_{min}$, when $\frac{\partial V}{\partial \phi} = 0$ If $\mu^2 > 0$, the situation for a massive particle, then $\phi = \phi_{min}$ when $\phi = 0$; this is the normal situation for the lowest energy vacuum state with $V = O$. However, if $\mu^2 < 0$ then

$$\phi = \phi_{min} \quad \text{when} \quad \phi = \pm v = \pm\sqrt{\frac{-\mu^2}{\lambda}} \tag{6}$$

Here the lowest energy state has ϕ finite, with $V = \frac{-\mu^4}{4\lambda}$, so that V is everywhere a non-zero constant. The quantity v is called the vacuum expectation value of the scalar boson field ϕ. Figure 1 shows V as a function of ϕ, both

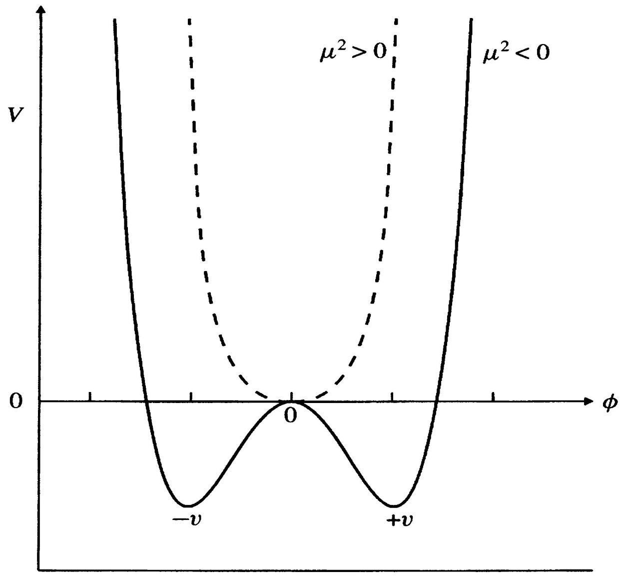

Figure 1: Plot of the potential V in (5) as a function of a one-dimensional scalar field ϕ for the two cases $\mu^2 > 0$ and $\mu^2 < 0$.

for $\mu^2 > 0$ and $\mu^2 < O$. In $\mu^2 < O$ case there are two minima, $\phi_{min} = +v$ and $-v$. In weak interactions, we are however concerned with evaluating small perturbations about the energy minimum, so that we should expand the field variable ϕ, not about zero but about the chosen vacuum minimum v (or $-v$), i.e.

$$\phi = v + \sigma(x) \tag{7}$$

where $\sigma(x)$ is the (variable) value of the field over and above the constant and uniform value, v. Substituting in (5) we get:

$$L = \frac{1}{2}(\partial_\mu \sigma)^2 - \lambda v^2 \sigma^2 - (\lambda v \sigma^3 + \frac{1}{4}\lambda \sigma^4) + \text{constant} \tag{8}$$

where the constant refers to terms in v^2 and v^4 and the third term on the right-hand side represents the interaction of the σ field with itself. The first two terms on the right will be the same for either value of v, and when compared with (4), suggest that the term $-\lambda v^2 \sigma^2$ is a mass term, with the positive value

$$m = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2} \tag{9}$$

So, by making a perturbation expansion about either of the two minima $\pm v$, a real positive mass - as against an imaginary one in (6) - has appeared. The perturbation expansion must be made about one or other of the two minima - chosen for example by the toss of a coin - but when this is done, of course the symmetry in Figure 1 will be broken. This behaviour is called *spontaneous symmetry breaking.* Many examples exist in physics. A bar magnet heated above the Curie point has its elementary magnetic domains pointed in random directions, with zero net moment, and the Lagrangian is invariant under rotations of the magnet in space. On cooling, the domains will set in a particular direction, that of the resultant moment, and the rotational symmetry is spontaneously broken. [1, § 8.12] [4, § 4.1]

3 The Linear Sigma Model

Now we are consider that the broken symmetry is continuous. A generalization of the preceding theory called the linear sigma model and it is the most important example of this topic.

The Lagrangian of the linear sigma model involves a set of N real scalar field $\phi^i(x)$

$$L = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi)^2]^2 \tag{10}$$

with an implicit sum over i in each factor $(\phi^i)^2$. The above Lagrangian is invariant under the symmetry

$$\phi^i \to R^{ij}\phi^j \tag{11}$$

for any $N \times N$ orthogonal matrix R. The group of transformations (11) is just the rotation group in N dimensions, also called the N-dimensional orthogonal group or simply O(N).

Again the lowest-energy classical configuration is a constant field ϕ_0^i, whose value is chosen to minimize the potential

$$L = -\tfrac{1}{2}\mu^2(\phi^i)^2 + \tfrac{\lambda}{4}[(\phi)^2]^2$$

(see Figure 2). This potential is minimized for any ϕ_0^i that satisfies

$$(\phi_0^i)^2 = \tfrac{\mu^2}{\lambda}.$$

This condition determines only the length of the vector ϕ_0^i; its direction is arbitrary. It is conventional to choose coordinates so that ϕ_0^i points in the Nth direction

$$\phi_0^i = (0, 0, ..., 0, v), \quad \text{where} \quad v = \frac{\mu}{\sqrt{\lambda}}. \tag{12}$$

We can now define a set of shifted fields by writing

$$\phi^i(x) = (\pi^k(x), v + \sigma(x)), k = 1, ..., N-1. \tag{13}$$

It is now straightforward to rewrite the Lagrangian (10) in terms of the π and σ fields. The result is

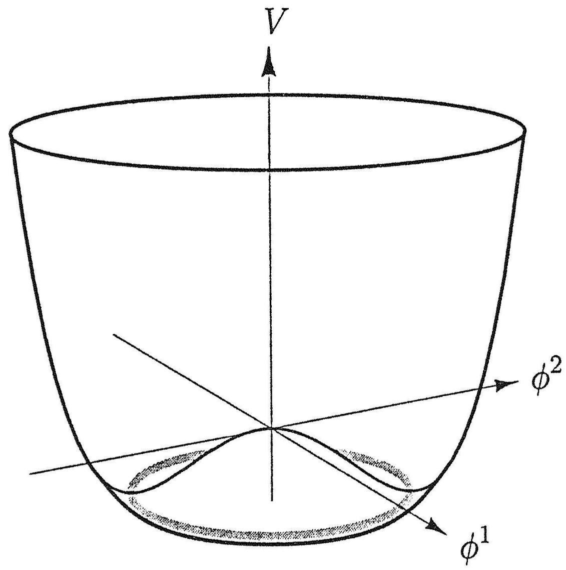

Figure 2: Potential for spontaneous breaking of a continuous O(N) symmetry, drawn for the case $N = 2$. Oscillations along the trough in the potential correspond to the massless π fields.

$$\begin{aligned} L = \frac{1}{2}(\partial_\mu \pi^k)^2 \quad &+ \quad \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\pi^k)^2\sigma - \\ &- \quad \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}[(\pi^k)^2]^2. \end{aligned} \tag{14}$$

We obtain a massive σ field just as in (8), and also a set of N-1 massless π fields. The original O(N) symmetry is hidden, leaving only the subgroup O(N-1), which rotates the π fields among themselves. Referring to Figure 2, we note that the massive σ field describes oscillations of ϕ^i in the radial direction, in which the potential has a nonvanishing second derivative. The massless π fields describe oscillations of ϕ^i in the tangential directions, along the trough of the potential. The trough is an (N-1) dimensional surface, and all N-1 directions are equivalent, reflecting the unbroken O(N-1) symmetry. [2, § 11.1]

4 Goldstone's Theorem

The appearance of massless particles when a continuous symmetry is spontaneously broken is a general result, known as Goldstone's theorem. To state the theorem precisely, we must count the number of linearly independent continuous symmetry transformations. In the linear sigma model, there are no continuous symmetries for N=1, while for N=2 there is a single direction of rotation. A rotation in N dimensions can be in any one of N(N-1)/2 planes, so the O(N)-symmetric theory has N(N-1)/2 continuous symmetries. After spontaneous symmetry breaking there are (N-1)(N-2)/2 remaining symmetries, corresponding to rotations of the (N-1) π fields. The number of broken symmetries is the difference, N-1.

Goldstone's theorem states that for every spontaneously broken continuous symmetry, the theory must contain a massless particle. We have just seen that this theorem holds in the linear sigma model, at least at the classical level. The massless fields that arise through spontaneous symmetry breaking are called Goldstone bosons. Many light bosons seen in physics, such as the pions, may be interpreted (at least approximately) as Goldstone bosons.

Consider, then, a theory involving several fields $\phi^a(x)$, with a Lagrangian of the form

$$L = T - V(\phi) \tag{15}$$

Let ϕ_0^a be a constant field that minimizes V, so that

$$\frac{\partial}{\partial \phi^a} V|_{\phi^a(x)=\phi_0^a} = 0. \tag{16}$$

Expanding V about this minimum, we find

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^a(\phi - \phi_0)^b(\frac{\partial^2}{\partial\phi^a\partial\phi^b})V)_{\phi_0} + ... \tag{17}$$

The coefficient of the quadratic term,

$$(\frac{\partial^2}{\partial\phi^a\partial\phi^b})V)_{\phi_0} = m_{ab}^2, \tag{18}$$

is a symmetric matrix whose eigenvalues give the masses of the fields. These eigenvalues cannot be negative, since ϕ_0 is a minimum. To prove Goldstone's theorem, we must show that every continuous symmetry of the Lagrangian (15) that is not a symmetry of ϕ_0 gives rise to a zero eigenvalue of this mass matrix.

A general continuous symmetry transformation has the form

$$\phi^a \rightarrow \phi^a + \alpha\Delta^a(\phi), \tag{19}$$

where α is an infinitesimal parameter and Δ^a is some function of all the ϕ's. Specialize to constant fields; then the derivative terms in L vanish and the potential alone must be invariant under (19). This condition can be written

$$V(\phi^a) = V(\phi^a + \alpha\Delta^a(\phi)) \qquad \text{or} \qquad \Delta^a(\phi)\frac{\partial}{\partial\phi^a V(\phi)} = 0. \tag{20}$$

Now differentiate with respect to ϕ^b, and set $\phi = \phi_0$:

$$0 = (\frac{\partial \Delta^a}{\partial \phi^b})_{\phi_0} (\frac{\partial V}{\partial \phi^a})_{\phi_0} + \Delta^a(\phi_0)(\frac{\partial^2}{\partial \phi^a \partial \phi^b})V)_{\phi_0}. \tag{21}$$

The first term vanishes since ϕ_0 is a minimum of V, so the second term must also vanish. If the transformation leaves ϕ_0 unchanged (i.e., if the symmetry is respected by the ground state), then $\Delta^a(\phi_0) = 0$ and this relation is trivial. A spontaneously broken symmetry is precisely one for which $\Delta^a(\phi_0) \neq 0$; in this case $\Delta^a(\phi_0)$ is our desired vector with eigenvalue zero, so Goldstone's theorem is proved. [2, § 11.1]

References

[1] D. H. Perkins. *Introduction to high energy physics.* Cambridge University Press, 4 edition, 2000.

[2] M. E. Peskin and D. V. Schroeder. *Introduction to quantum field theory.* Addison-Wesley Publishing Company, 1995.

[3] F. Strocchi. *Symmetry Breaking.* Springer, Berlin Heidelberg, 2 edition, 2008.

[4] A. Zee. *Quantum field theory in a nutshell.* Princeton University Press, 2003.