Carter-Penrose diagrams and black holes

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Seminar Structure

Idea

- Curved spacetime manifolds can be often approximated by manifolds with high degrees of symmetry
- It would be useful to be able to draw spacetimes diagrams that capture global properities and casual structure of sufficiently symmetric spacetimes
- We need to do a conformal transformation which brings entire manifold onto a compact region such that we can fit the spacetime on a piece of paper.

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Minkowski space

- The Minkowski metric in polar coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \quad (1)$$

where $d\Omega^2 = d\Theta^2 + sin^2\Theta d\Phi^2$ is a metric on a unit two-sphere.

$$-\infty < t < \infty, \; 0 \leq r < \infty. \quad (2)$$

- We need coordinates with finite ranges - at first switch to null coordinates:

$$u = t - r, \; v = t + r \quad (3)$$

with corresponding ranges given by:

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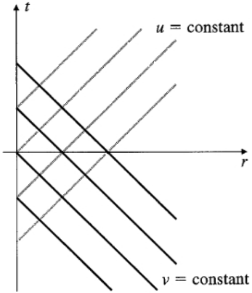

Null radial coordinates on Minkowski space.

Each point represents a 2-sphere of radius $r = \frac{1}{2}(v - u)$. The Minkowski metric in null coordinates is given by

$$ds^2 = -\frac{1}{2}(dudv + dvdu) + \frac{1}{4}(v - u)^2 d\Omega^2. \quad (5)$$

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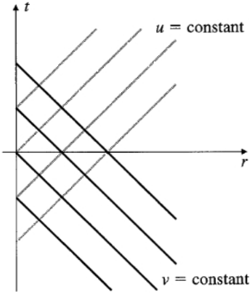

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calculating...

Using the arctangent we bring infinity into a finite coordinate value:

$$U = \arctan(u),\; V = \arctan(v), \tag{6}$$

with ranges

$$-\pi/2 < U < \pi/2,\; -\pi/2 < V < \pi/2,\; U \leq V. \tag{7}$$

After easy calculations one gets that the metric given in (5) in this coordinates takes form:

$$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\left[-2(dUdV + dVdU) + \sin^2(V-U)d\Omega^2\right]. \tag{8}$$

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Transforming back to the timelike coordinate T and radial coordinate R:

$$T = U + V,\ R = V - U \tag{9}$$

with ranges

$$0 \leq R < \pi,\ |T| + R < \pi. \tag{10}$$

Now the metric is:

$$ds^2 = \omega^{-2}(T,R)\left(-dT^2 + dR^2 + sin^2 R d\Omega^2\right). \tag{11}$$

where

$$\omega(T,R) = 2\cos U \cos V = \cos T + \cos R. \tag{12}$$

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$$\tilde{ds}^2 = \omega^2(T, R)ds^2 = -dT^2 + dR^2 + \sin^2 R d\Omega^2. \qquad (13)$$

This describes the manifold $\mathbf{R} \times S^3$, where 3-sphere is maximally symmetric and static.

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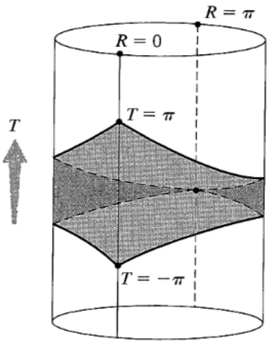
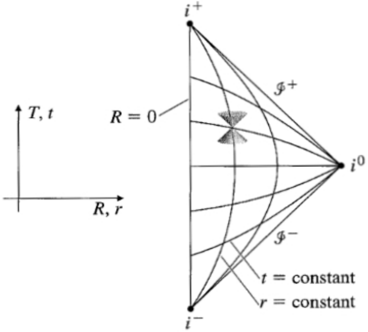
Einstein static universe



The Einstein static universe, $\mathbf{R} \times S^3$, portrayed as a cylinder. The shaded region is conformally related to Minkowski space.

Conformal diagram of Minkowski space



The conformal diagram of Minkowski space. Light cones are at ±45° throughout the diagram.

...

The structure of the conformal diagram allows us to subdivide conformal infinity into a few different regions:

$$
\begin{aligned}
i^+ &= \text{future timelike infinity } (T = \pi \ , \ R = 0) \\
i^0 &= \text{spatial infinity } (T = 0 \ , \ R = \pi) \\
i^- &= \text{past timelike infinity } (T = -\pi \ , \ R = 0) \\
\mathcal{I}^+ &= \text{future null infinity } (T = \pi - R \ , \ 0 < R < \pi) \\
\mathcal{I}^- &= \text{past null infinity } (T = -\pi + R \ , \ 0 < R < \pi)
\end{aligned}
$$

Note that i^+, i^0, and i^- are actually points, since $R = 0$ and $R = \pi$ are the north and south poles of S^3. Meanwhile $\mathcal{I}^+$ and $\mathcal{I}^-$ are actually null surfaces, with the topology of $\times S^2$.

...

The conformal diagram for Minkowski spacetime contains a number of important features: radial null geodesics are at the $\pm 45^\circ$ in the diagram. All timelike geodesics begin at i^- and end at i^+. All null geodesics begin at $\mathcal{I}^-$ and end at $\mathcal{I}^+$; all spacelike geodesics both begin and end at i^0. On the other hand, there can be non-geodesic timelike curves that end at null infinity (if they become "asymptotically null").

Example

- When we put polar coordinates on space, the metric becomes:

$$ds^2 = -dt^2 + t^{2q}\left(dr^2 + r^2 d\Omega^2\right) \tag{14}$$

with $0 < q < 1$.

- Crucial difference between this metric and that of Minkowski space: singularity at $t = 0$ - it restricts the range of coordinates:

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Example

We choose new time coordinate η called conformal time, which satisfies:

$$dt^2 = t^{2q} d\eta^2 \tag{16}$$

with range same as of t:

$$0 < \eta < \infty. \tag{17}$$

This allows as to bring out the scale factor as an overall conformal factor times Minkowski:

$$ds^2 = [(1-q)\eta]^{2q/(1-q)} \left(-d\eta^2 + dr^2 + r^2 d\Omega^2\right) \tag{18}$$

...

After the same sequence of coordinate transformations as previously, one gets (η, r) to (T, R) with ranges:

$$0 \leq R,\ 0 < T,\ T + R < \pi. \tag{19}$$

the metric (18) becomes:

$$ds^2 = \omega^{-2}(T, R)\left(-dT^2 + dR^2 + sin^2 R d\Omega^2\right) \tag{20}$$

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- Once again we expressed our metric as a conformal factor times that of the Einstein static universe.
- Difference between this case and that of flat spacetime: timelike coordinate ends at singularity $T = 0$

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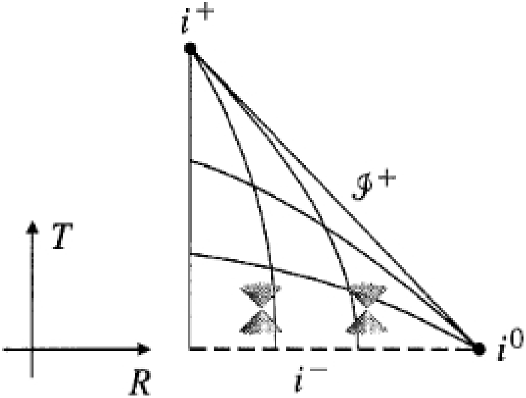
Conformal diagram for a Robertson=Walker universe



More complicated spacetimes: black holes

- The conformal diagram gives us an idea of the casual structure of the spacetime, e.g. whether the past or future light cones of two specified points intersect.
- In Minkowski space this is always true for any two points. Curved spacetimes - more interesting.

More complicated spacetimes: black holes

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Very brief introduction

- Schwarzschild solution describes spherically symmetric vacuum spacetimes.
- Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (21)$$

- This is true for any spherically symmetric vacuum solution to Einstein's equations; M functions as a parameter. Note that as $M \to 0$ we recover Minkowski space, which is to be expected. Note also that the metric becomes progressively Minkowskian as we go to $r \to \infty$; this property is known as **asymptotic flatness**.

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Very brief introduction

One way of understanding a geometry is to explore its causal structure, as defined by the light cones. We therefore consider radial null curves, those for which θ and ϕ are constant and $ds^2 = 0$:

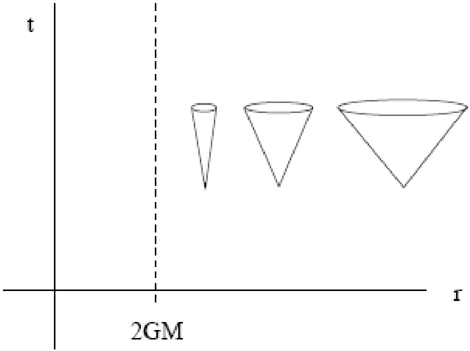
$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2, \qquad (22)$$

from which we see that

$$\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1} \qquad (23)$$

This measures the slope of the light cones on a spacetime diagram of the t-r plane. For large r the slope is ± 1, as it would be in flat space, while as we approach $r = 2GM$ we get $dt/dr \to \pm\infty$, and the light cones 'close up'

Very brief introduction



A light ray which approaches $r = 2GM$ never seems to get there, at least in this coordinate system; instead it seems to asymptote to this radius.

Very brief introduction

The problem with our current coordinates is that $dt/dr \rightarrow \infty$ along radial null geodesics which approach $r = 2GM$; progress in the r direction becomes slower and slower with respect to the coordinate time t. We suspect that our coordinates may not have been good for the entire manifold .

Fixing $r = 2GM$: Eddington-Fielkenstein coordinates

- By changing coordinate t to the new one $\tilde{u}$, which has the nice property that if we decrease r along a radial curve null curve $\tilde{u} =$ constant, we go right through the event horizon $r = 2GM$ without any problems.
- The region $r \leq 2GM$ is now included in our spacetime, since physical particles can easily reach there and pass through.
- Still, there are other directions in which we can extend our manifold.

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Kruskal coordinates

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dv^2 + du^2) + r^2 d\Omega^2 \qquad (24)$$

where r is defined implicitly from

$$(u^2 - v^2) = \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \qquad (25)$$

Penrose diagram for a Schwarzschild black hole

We start with the null version of the Kruskal coordinates, in which the metric takes the form

$$ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM}(du'dv' + dv'du') + r^2 d\Omega^2 \; , \qquad (26)$$

where r is defined implicitly via

$$u'v' = \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \; . \qquad (27)$$

Penrose diagram for a Schwarzschild black hole

Then essentially the same transformation as was used in flat spacetime suffices to bring infinity into finite coordinate values:

$$u'' = \arctan\left(\frac{u'}{\sqrt{2GM}}\right), \; v'' = \arctan\left(\frac{v'}{\sqrt{2GM}}\right), \qquad (28)$$

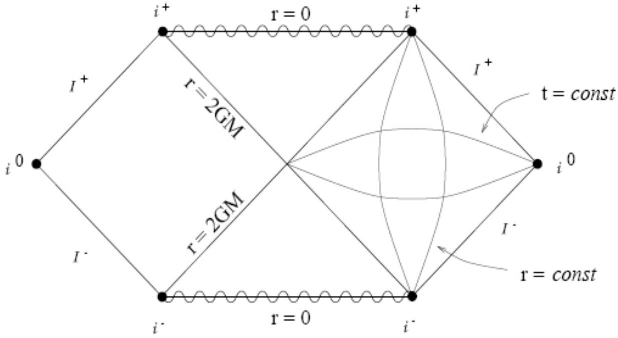
with ranges

$$-\pi/2 < u'' < +\pi/2, \; -\pi/2 < v'' < +\pi/2, \; -\pi < u'' + v'' < \pi \; . \qquad (29)$$

Penrose diagram for a Schwarzschild black hole

The (u'', v'') part of the metric (that is, at constant angular coordinates) is now conformally related to Minkowski space. In the new coordinates the singularities at $r = 0$ are straight lines that stretch from timelike infinity in one asymptotic region to timelike infinity in the other.

Penrose diagram for a Schwarzschild black hole



The Penrose diagram for the maximally extended Schwarzschild solution

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Thanks for listening