Carter-Penrose diagrams and black holes

Ewa Felinska

The basic introduction to the method of building Penrose diagrams has been presented, starting with obtaining a Penrose diagram from Minkowski space. An example of computation for curved spacetime has been provided, with a conformal diagram for Robertson-Walker universe. A way of further coordinate transformations needed to extend given manifold has been provided and a Penrose diagram for Schwarzschild black hole has been constructed.

1. Penrose diagrams

1.1. Obtaining Penrose diagrams from Minkowski space

Curved spacetime manifolds can be often approximated by manifolds with high degrees of symmetry. It would be useful to be able to draw spacetimes diagrams that capture global properities and casual structure of sufficiently symmetric spacetimes. What is needed to be done is a conformal transformation which brings entire manifold onto a compact region such that we can fit the spacetime (ie. its infinities) on a finite 2-dimensional diagram, known as Penrose-Carter diagram (or Carter-Penrose diagram or just conformal diagram).

Let us start with the Minkowski space, with metric in polar coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{1}$$

where $d\Omega^2 = d\Theta^2 + sin^2\Theta d\Phi^2$ is a metric on a unit two-sphere and ranges of timelike and spacelike coordinates are:

$$-\infty < t < \infty,\ 0 \leq r < \infty. \tag{2}$$

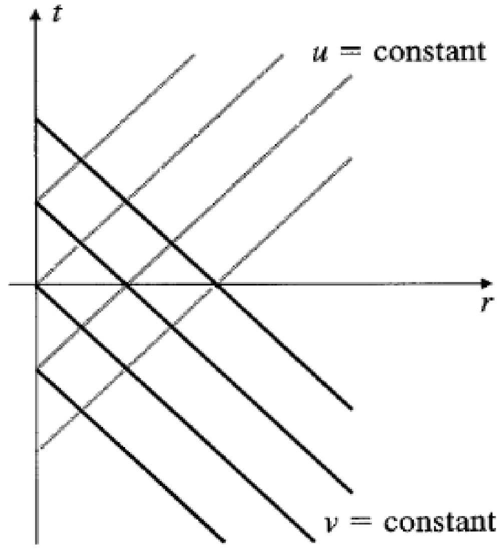
In order to get coordinates with finite ranges, let us switch to null coordinates:

$$u = t - r,\ v = t + r \tag{3}$$

with corresponding ranges (Fig.1.):

$$-\infty < u < \infty,\ -\infty < v < \infty,\ u \leq v. \tag{4}$$

Fig. 1. Each point represents a 2-sphere of radius $r = \frac{1}{2}(v - u)$.



The Minkowski metric in null coordinates is given by

$$ds^2 = -\frac{1}{2}(dudv + dvdu) + \frac{1}{4}(v - u)^2 d\Omega^2. \tag{5}$$

We can bring infinity into a finite coordinate value by using the arctangent:

$$U = \arctan(u),\, V = \arctan(v), \tag{6}$$

with ranges

$$-\pi/2 < U < \pi/2,\ -\pi/2 < V < \pi/2,\, U \leq V. \tag{7}$$

After easy calculations one gets that the metric given in (5) in this coordinates takes form:

$$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\left[-2(dUdV + dVdU) + \sin^2(V - U)d\Omega^2\right]. \tag{8}$$

Transforming back to the timelike coordinate T and radial coordinate R:

$$T = U + V,\, R = V - U \tag{9}$$

with finite ranges

$$0 \leq R < \pi,\, |T| + R < \pi, \tag{10}$$

the metric is given by:

$$ds^2 = \omega^{-2}(T,R)\left(-dT^2 + dR^2 + sin^2 Rd\Omega^2\right). \quad (11)$$

where

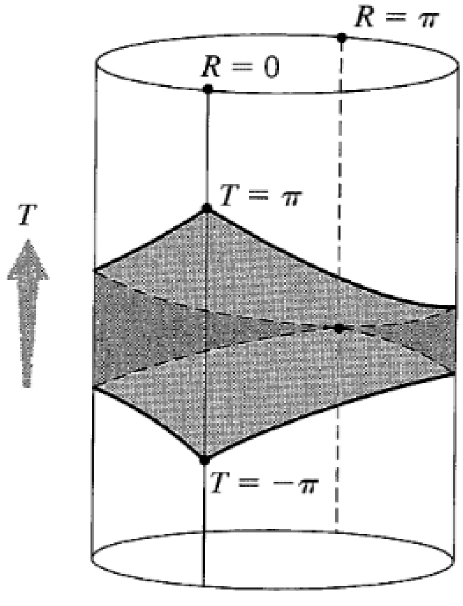
$$\omega(T,R) = 2\cos U \cos V = \cos T + \cos R. \quad (12)$$

Finally, the original Minkowski metric, which we denoted ds^2, is related by a conformal transformation to the new metric:

$$\tilde{ds}^2 = \omega^2(T,R)ds^2 = -dT^2 + dR^2 + \sin^2 Rd\Omega^2. \quad (13)$$

This describes the manifold $\mathbf{R} \times S^3$, where 3-sphere is maximally symmetric and static.

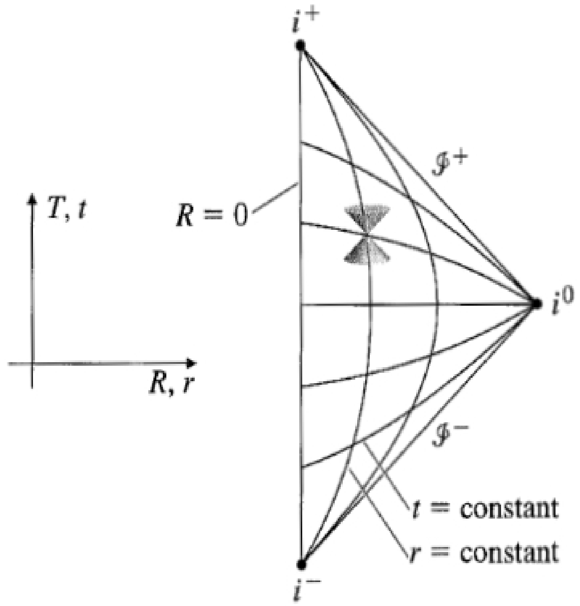
Fig. 2. The Einstein static universe, $\mathbf{R} \times S^3$, portrayed as a cylinder. The shaded region is conformally related to Minkowski space (see: Fig.3.).



The structure of the conformal diagram allows us to subdivide conformal infinity into a few different regions (Fig.2.):

i^+ = future timelike infinity ($T = \pi$, $R = 0$)
i^0 = spatial infinity ($T = 0$, $R = \pi$)
i^- = past timelike infinity ($T = -\pi$, $R = 0$)
$\mathcal{I}^+$ = future null infinity ($T = \pi - R$, $0 < R < \pi$)
$\mathcal{I}^-$ = past null infinity ($T = -\pi + R$, $0 < R < \pi$)

Fig. 3. The conformal diagram of Minkowski space. Light cones are ±45° throughout the diagram.



Note that i^+, i^0, and i^- are actually points, since $R = 0$ and $R = \pi$ are the north and south poles of S^3. Meanwhile, $\mathcal{I}^+$ and $\mathcal{I}^-$ are null surfaces, with the topology of $\mathbf{R} \times S^2$.

The conformal diagram for Minkowski spacetime contains a number of important features: radial null geodesics are at the ±45° angle in the diagram. All timelike geodesics begin at i^- and end at i^+. All null geodesics begin at $\mathcal{I}^-$ and end at $\mathcal{I}^+$; all spacelike geodesics both begin and end at i^0. On the other hand, there can be non-geodesic timelike curves that end at null infinity (if they become "asymptotically null").

1.2. Examples

When we put polar coordinates on space, the metric becomes:

$$ds^2 = -dt^2 + t^{2q}\left(dr^2 + r^2 d\Omega^2\right) \tag{14}$$

with $0 < q < 1$. The crucial difference between this metric and the one of Minkowski space is a singularity at $t = 0$, what restricts the range of coordinates:

$$0 < t < \infty,\ 0 \leq r < \infty. \tag{15}$$

We choose new time coordinate η called conformal time, which satisfies:

$$dt^2 = t^{2q} d\eta^2 \tag{16}$$

with range same as of t:

$$0 < \eta < \infty. \tag{17}$$

This allows us to bring out the scale factor as an overall conformal factor times Minkowski:

$$ds^2 = [(1-q)\eta]^{2q/(1-q)} \left(-d\eta^2 + dr^2 + r^2 d\Omega^2\right) \tag{18}$$

After the same sequence of coordinate transformations as previously, one gets (η, r) to (T, R) with ranges:

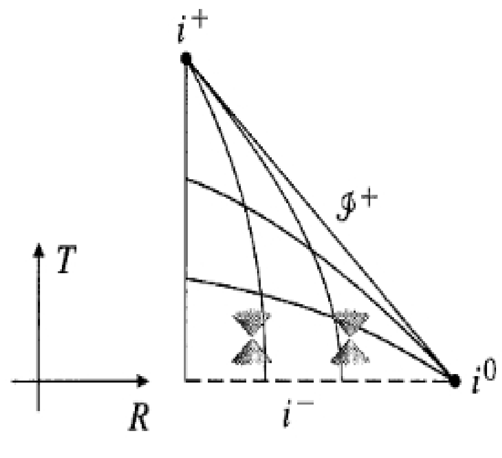
$$0 \leq R,\ 0 < T,\ T + R < \pi. \tag{19}$$

The metric (18) becomes:

$$ds^2 = \omega^{-2}(T, R) \left(-dT^2 + dR^2 + sin^2 R d\Omega^2\right) \tag{20}$$

Once again we expressed our metric as a conformal factor times that of the Einstein static universe. The difference between this case and that of flat spacetime is that timelike coordinate ends at singularity $T = 0$

Fig. 4. Conformal diagram for a Robertson-Walker universe.



2. Black holes

The conformal diagram gives us an idea of the casual structure of the spacetime, e.g. whether the past or future light cones of two specified points intersect. In Minkowski space this is always true for any two point, but the situation becomes much more interesting in curved spacetimes. A good example is Schwarzschild solution, which describes spherically symmetric vacuum spacetimes.

Schwarzschild metric is given by:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{21}$$

This is true for any spherically symmetric vacuum solution to Einstein's equations; M functions as a parameter. Note that as $M \to 0$ we recover Minkowski space, which is to be expected. It is also worth noting, that the metric becomes progressively Minkowskian as we go to $r \to \infty$; this property is known as **asymptotic flatness**.

One way of understanding a geometry is to explore its causal structure, as defined by the light cones. We therefore consider radial null curves, those for which θ and ϕ are constant and $ds^2 = 0$:

$$ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2, \tag{22}$$

from which we see that

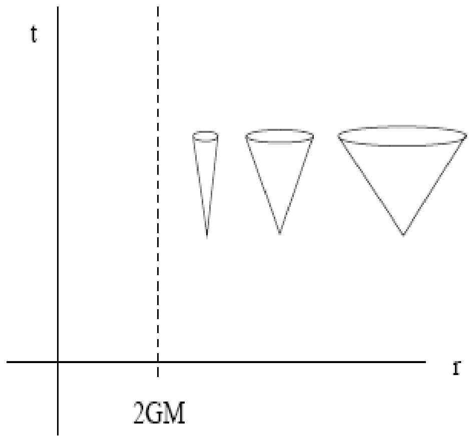
$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1} \tag{23}$$

This measures the slope of the light cones on a spacetime diagram of the t-r plane. For large r the slope is ± 1, as it would be in flat space, while as we approach $r = 2GM$ we get $dt/dr \to \pm\infty$, and the light cones 'close up'

The problem with our current coordinates is that progress in the r direction becomes slower and slower with respect to the coordinate time t. We suspect that our coordinates may not have been good for the entire manifold. Thus, lets have a closer look at EddingtonFinkelstein coordinates, a pair of coordinate systems which are adapted to radial null geodesics for a Schwarzschild geometry.

By changing coordinate t to the new one $\tilde{u}$, which has the nice property that if we decrease r along a radial curve null curve $\tilde{u} =$ constant, we go right through the event horizon $r = 2GM$ without any problems. The region $r \leq 2GM$ is now included in our spacetime, since physical particles can easily reach there and pass through - the apparent singularity at the Schwarzschild radius is not a physical singularity but only a coordinate one. Still, there are other directions in which we can extend our manifold.

Fig. 5. Light ray which approaches $r = 2GM$ never seems to get there, at least in this coordinate system; instead it seems to asymptote to this radius.



3. Penrose diagrams for Schwarzschild black holes

Kruskal coordinates

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dv^2 + du^2) + r^2 d\Omega^2 \tag{24}$$

where r is defined implicitly from

$$(u^2 - v^2) = \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \tag{25}$$

We start with the null version of the Kruskal coordinates, in which the metric takes the form

$$ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM}(du'dv' + dv'du') + r^2 d\Omega^2 \ , \tag{26}$$

where r is defined implicitly via

$$u'v' = \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \ . \tag{27}$$

Then essentially the same transformation as was used in flat spacetime suffices to bring infinity into finite coordinate values:

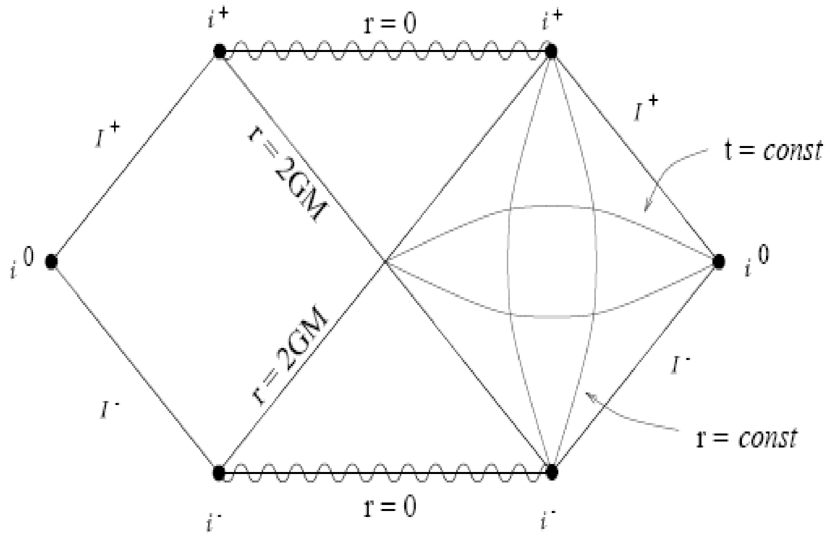
$$u'' = \arctan\left(\frac{u'}{\sqrt{2GM}}\right), \ v'' = \arctan\left(\frac{v'}{\sqrt{2GM}}\right), \tag{28}$$

with ranges

$$-\pi/2 < u'' < +\pi/2,\ -\pi/2 < v'' < +\pi/2,\ -\pi < u'' + v'' < \pi\ . \qquad (29)$$

The (u'', v'') part of the metric (that is, at constant angular coordinates) is now conformally related to Minkowski space. In the new coordinates the singularities at $r = 0$ are straight lines that stretch from timelike infinity in one asymptotic region to timelike infinity in the other.

Fig. 6. Penrose diagrams for Schwarzschild black holes



4. Conclusions

Penrose diagrams capture the causal relations between different points in spacetime, with the conformal factor chosen in a way that entire infinite spacetime is transformed into a diagram of finite size. This gives a very intuitive picture of the whole spacetime and its signularities, thus while comparing different spacetimes it is much easier to compare their conformal diagrams than metrics themselves.

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