Uncertainty relation for time and energy

University of Wrocław

Marcin Gonera

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Seminar structure

- algebraical derivation of Robertson relation
- interpretation (*Einstein's box* thought experiment)
- Lagrangian formalism of an object with proper time as dynamic variable
- Hamiltonian formalism analysis
- quantization

Robertson relation

For any two operators *A* and *B* and a state $|\psi\rangle$ we have

$$\langle\psi|A^\dagger A|\psi\rangle\langle\psi|B^\dagger B|\psi\rangle = \| A|\psi\rangle\|^2 \| B|\psi\rangle\|^2 \geq \left|\langle\psi|A^\dagger B|\psi\rangle\right|^2$$

On the other hand,

$$\left|\langle\psi|A\,B|\psi\rangle\right|^2 \geq \left|\,\mathrm{Im}\langle\psi|A\,B|\psi\rangle\right|^2 = \left|\frac{1}{2\,\mathrm{i}}\langle\psi|A\,B - B^\dagger A^\dagger|\psi\rangle\right|^2$$

For Hermitian operators we get

$$\langle A^2\rangle\langle B^2\rangle \geq \frac{1}{4}\left|\langle[A,B]\rangle\right|^2$$

$$\boxed{\Delta A\,\Delta B \geq \frac{1}{2}\left|\langle[A,B]\rangle\right|}$$

where $\langle X\rangle \equiv \langle\psi|X|\psi\rangle\,,\ \Delta X \equiv \sqrt{\langle(X-\langle X\rangle)^2\rangle}$

Uncertainty principles

The Robertson inequality can be used for deriving uncertainty relations for any two observables which do not commute, like position and corresponding momentum coordinate:

$$[x_i, p_i] = \mathrm{i}\hbar \quad \Rightarrow \quad \Delta x_i \Delta p_i \geq \frac{\hbar}{2}$$

There is one uncertainty relation which is not so obvious consequence of the Robertson relation: the time-energy uncertainty principle. It was clear to many founders of quantum mechanics that the following relation holds:

$$\boxed{\Delta t \Delta E \geq \frac{\hbar}{2}}$$

but it was not clear what Δt was, because the time at which a particle has given state is not an operator belonging to the particle, it is a parameter describing the evolution of the system.

Interpretation

The interpretation of Δt depends on the kind of experiment. It can be the lifetime of a state, but usually it is the accuracy of time measurement.

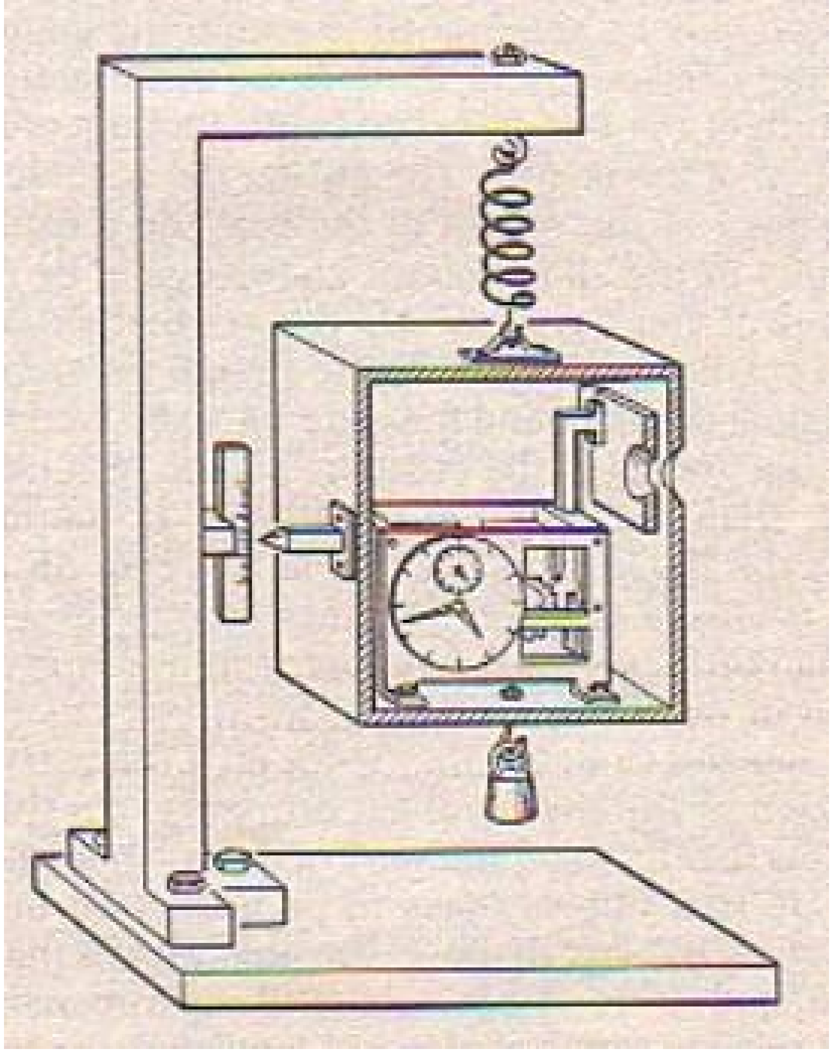
One false formulation is that measuring the energy of a quantum state to accuracy ΔE requires a time interval $\hbar/2\Delta E$:

$$\Delta t \geq \frac{\hbar}{2\Delta E}$$

Such formulation, with Δt as duration of measurement, is not always true!

Einstein's box

Consider a box filled with light. The box has a hole in one of the walls and a shutter, which opens and quickly closes the hole, such that some of the light escapes. There is a clock, which can be set such that the moment at which the photon escapes is known. In order to measure the energy of the leaving photon, Einstein proposed weighing the box before and after the emission – the box can be suspended on a spring, there is a pointer and a scale. The difference of masses multiplied by c^2 will equal the energy of the photon.



Einstein's box

The idea of this thought experiment was that the uncertainty of time, at which the photon escapes, can be as small as one wishes

$$\Delta t \;\rightarrow\; 0$$

and the energy of photon can be measured to finite accuracy, such that

$$\Delta t \, \Delta E \;\rightarrow\; 0$$

what is in contradiction with the time-energy uncertainty relation.

Bohr realized that since the box is immersed in a gravitational field, then the uncertainty in position of the box alters the ticking rate of the clock.

Apart from this, a photon as an localized object with definite energy can not exist!

Lagrangian formalism

In the exact form, the relation is between proper time and rest mass of an object. We will select the simplest Lagrangian, which describes the Einstein's box and other systems of that kind (a massive object in a field):

$$L_0 = -m\,c\sqrt{-g_{\mu\nu}(x)\,\dot{x}^{\mu}\,\dot{x}^{\nu}} + e\,A_{\mu}(x)\,\dot{x}^{\mu}$$

where we assumed that gravitational ($g_{\mu\nu}$) and electromagnetic (A_{μ}) fields are given, and the variables are x^{μ}.

It is obvious that, for a clock, the proper time is a measurable quantity, so we have to find another Lagrangian which includes the proper time as additional dynamic variable. We consider

$$L = M\,(\dot{\tau} - \sqrt{-g_{\mu\nu}(x)\,\dot{x}^{\mu}\,\dot{x}^{\nu}}/c) + e\,A_{\mu}(x)\,\dot{x}^{\mu}$$

Now the dynamic variables are τ, M and x^{μ}.

Lagrangian formalism

Calculating equations of motion it can be checked that τ can be indentified with proper time of the object, and, in order to have the same equation as one derived from L_0, M must be identified with the constant mc^2.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{\tau}} = \frac{\partial L}{\partial \tau} \quad \Rightarrow \quad \dot{M} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{M}} = \frac{\partial L}{\partial M} \quad \Rightarrow \quad \dot{\tau} = \sqrt{-g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}}/c$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^{\rho}} = \frac{\partial L}{\partial x^{\rho}} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{M}{c}\frac{g_{\rho\mu}(x)\dot{x}^{\mu}}{\sqrt{-g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}}} + e A_{\rho}(x)\right] -$$

$$- \frac{M}{c}\frac{g_{\mu\nu,\rho}\dot{x}^{\mu}\dot{x}^{\nu}}{2\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}} - e A_{\mu,\rho}(x)\dot{x}^{\mu} = 0$$

Lagrangian formalism

We assume for simplicity that the fields are static: functions $g_{\mu\nu}$ and A_μ depend only on x^1, x^2, x^3, and we have $g_{i0} = 0$. Then our Lagrangian can be written as

$$\boxed{L = M\left(\dot{\tau} - \sqrt{f(x)^2 - g_{ij}(x)\,\dot{x}^i\dot{x}^j/c^2}\right) + c\,e\,A_0(x) + e\,A_i(x)\,\dot{x}^i}$$

where f is defined by $g_{00} = -f^2$.

The dynamic variables are τ, M, x^i (i = 1, 2, 3) and the dot denotes differential with respect to t.

Lagrangian formalism

The momentums conjugate to those variables are:

$$p_\tau \equiv \frac{\partial L}{\partial \dot{\tau}} = M$$

$$p_M \equiv \frac{\partial L}{\partial \dot{M}} = 0$$

$$p_i \equiv \frac{\partial L}{\partial \dot{x}^i} = -M \frac{1}{2\sqrt{f^2 - g_{kj}\dot{x}^k \dot{x}^j / c^2}} \left(-\frac{g_{kj}}{c^2} (\delta_i^k \dot{x}^j + \dot{x}^k \delta_i^j) \right) + e A_k \delta_i^k =$$

$$= \frac{M}{c^2} \frac{g_{ij}\dot{x}^j}{\sqrt{f^2 - g_{jk}\dot{x}^j \dot{x}^k / c^2}} + e A_i$$

Hamiltonian formalism

We can calculate the Hamiltonian:

$$H_0 \equiv p_\tau \dot{\tau} + p_M \dot{M} + p_i \dot{x}^i - L =$$
$$= f \sqrt{M^2 + c^2 g^{ij}(p_i - e A_i)(p_j - e A_j)} - c e A_0$$

where the velocities were expressed by momenta.

In our case, however, there exist two constraints: $\phi_1 \equiv M - p_\tau = 0$ and $\phi_2 \equiv p_M = 0$. Therefore we have to consider the total Hamiltonian $H \equiv H_0 + u_1\phi_1 + u_2\phi_2$, where u_1 and u_2 are undetermined Lagrange's multipliers.

Hamiltonian formalism

These multipliers can be determined by consistency conditions. Time-derivatives of the constraints are defined by Poisson's brackets with the Hamiltonian and they must be weakly equal to zero:

$$\dot{\phi}_1 = \{\phi_1, H\} \approx 0 , \quad \dot{\phi}_2 = \{\phi_2, H\} \approx 0$$

where Poisson's bracket in our case is

$$\{f, g\} \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial q_i} \right) =$$

$$= \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial p_\tau} - \frac{\partial f}{\partial p_\tau} \frac{\partial g}{\partial \tau} + \frac{\partial f}{\partial M} \frac{\partial g}{\partial p_M} - \frac{\partial f}{\partial p_M} \frac{\partial g}{\partial M} + \frac{\partial f}{\partial x^k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial x^k}$$

and there is summation over *k*.

Hamiltonian formalism

Calculating the consistency conditions we get the multipliers:

$$u_1 = -\frac{f\,M}{\sqrt{M^2 + c^2 g^{ij}(p_i - e\,A_i)(p_j - e\,A_j)}} \quad \text{and} \quad u_2 = 0$$

which require the Hamiltonian to be

$$\boxed{H = H_0 - \frac{f\,M(M - p_\tau)}{\sqrt{M^2 + c^2 g^{ij}(p_i - e\,A_i)(p_j - e\,A_j)}}}$$

Hamiltonian formalism

Hamilton's equations of motion are as follows:

$$\dot{\tau} = \frac{\partial H}{\partial p_\tau} = \frac{f M}{\sqrt{M^2 + c^2 g^{ij}(p_i - e A_i)(p_j - e A_j)}} \ , \quad \dot{p}_\tau = -\frac{\partial H}{\partial \tau} = 0$$

$$\dot{M} = \frac{\partial H}{\partial p_M} = 0 \ , \quad \dot{p}_M = -\frac{\partial H}{\partial M} \approx 0$$

$$\dot{x}^i = \frac{\partial H}{\partial p_i} \approx \frac{\partial H_0}{\partial p_i} \ , \quad \dot{p}_i = -\frac{\partial H}{\partial x^i} \approx -\frac{\partial H_0}{\partial x^i}$$

Quantization

Dirac's bracket, for our case, is

$$\{A,B\}_{\mathrm{D}} = \{A,B\} + \{A,\phi_1\}\{\phi_2,B\} - \{A,\phi_2\}\{\phi_1,B\}$$

We can calculate Dirac's brackets between canonical variables τ, p_τ, M, p_M, x^i, p_i:

$$\{\tau,p_\tau\}_{\mathrm{D}} = \{\tau,p_\tau\} + \{\tau,M - p_\tau\}\{p_M,p_\tau\} - \{\tau,p_M\}\{M - p_\tau,p_\tau\} =$$
$$= 1 + (-1)\cdot 0 - 0\cdot 0 = 1$$

$$\{\tau,M\}_{\mathrm{D}} = \{\tau,M\} + \{\tau,M - p_\tau\}\{p_M,M\} - \{\tau,p_M\}\{M - p_\tau,M\} =$$
$$= 0 + (-1)\cdot(-1) - 0\cdot 0 = 1$$

$$\{x^i,p_j\}_{\mathrm{D}} = \delta^i_j \;, \quad \text{the others} = 0$$

Quantization

The following set of variables:

$$\phi_1 \equiv M - p_\tau\ ,\ \ \phi_2 \equiv p_M\ ,\ \ T \equiv \tau - p_M\ ,\ \ E \equiv p_\tau\ ,\ \ x^i\ ,\ \ p_i \quad (i = 1, 2, 3)$$

are canonical variables. The subset $\{T, E, x^i, p_i\}$ can be interpreted as canonical variables on the submanifold defined by the constraints $\phi_1 = 0$ and $\phi_2 = 0$.

Thus, on the submanifold, we have $M = p_\tau$ and $p_M = 0$, what gives $T = \tau$ and $E = M$ ($= mc^2$). Then the Dirac's brackets take form

$$\boxed{\{\tau , E\}_{\mathrm{D}} = 1 \ ,\quad \{x^i, p_j\}_{\mathrm{D}} = \delta^i_j \ ,\quad \text{the others} = 0}$$

Quantization

It follows from the above that the rest energy $E = mc^2$ is the general momentum conjugate to the proper time τ. If we quantize our system by Dirac's procedure, there are corresponding operators:

$$\hat{\tau},\ \hat{E},\ \hat{x}^i,\ \hat{p}_i\ \ (i = 1, 2, 3)$$

which satisfy following commutation relations:

$$[\hat{\tau}, \hat{E}] = [\hat{x}^i, \hat{p}_i] = \mathrm{i}\hbar$$

We can now substitute the first commutator into Robertson relation, and it leads to the following uncertainty relation:

$$\boxed{c^2 \Delta m \Delta \tau \geq \frac{\hbar}{2}}$$

When the velocity is small, the relation translates into $\Delta E \Delta t \geq \hbar/2$.

Bibliography

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