Uncertainty relation for time and energy

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Abstract. Algebraic derivation of Robertson relation is shown. Then the uncertainty relation for time and energy is discussed as not obvious consequence of Robertson relation. Finally, it is shown that the uncertainty relation for proper time and rest mass can be derived only if we describe the system by a Lagrangian which includes proper time as additional dynamic variable.

I. FUNDAMENTALS

For any two operators A and B and a state $|\psi\rangle$ we have

$$\langle\psi|A^\dagger A|\psi\rangle\langle\psi|B^\dagger B|\psi\rangle \;=\; \|A|\psi\rangle\|^2\|B|\psi\rangle\|^2 \;\geq\; \left|\langle\psi|A^\dagger B|\psi\rangle\right|^2 \tag{1}$$

where the inequality is the Cauchy-Schwarz statement for inner product of the two vectors $A|\psi\rangle$ and $B|\psi\rangle$. On the other hand, the expectation value of the product AB is greater than the magnitude of its imaginary part:

$$\left|\langle\psi|AB|\psi\rangle\right|^2 \;\geq\; \left|\,\mathrm{Im}\langle\psi|AB|\psi\rangle\right|^2 \;=\; \left|\frac{1}{2\mathrm{i}}\langle\psi|AB - B^\dagger A^\dagger|\psi\rangle\right|^2 \tag{2}$$

Putting these two inequalities together we get, for Hermitian operators,

$$\langle A^2\rangle\langle B^2\rangle \;\geq\; \frac{1}{4}\left|\langle[A,B]\rangle\right|^2 \tag{3}$$

where $\langle X\rangle \equiv \langle\psi|X|\psi\rangle$. We can shift these operators by their expectation values (then the commutator does not change) and take square root side by side, what gives a form of the Robertson relation:

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A, B] \rangle\right| \tag{4}$$

where the standard deviation of an observable X is defined as

$$\Delta X \equiv \sqrt{\langle (X - \langle X \rangle)^2 \rangle} \tag{5}$$

The inequality with the commutator term only was developed in 1929 by H.P. Robertson [12], and a little later E. Schrödinger added an anticommutator term.

The Robertson inequality (4) can be used for deriving uncertainty relations for any two observables which do not commute. For example, the commutator between position coordinate x_i and corresponding momentum coordinate p_i is equal to i$\hbar$, what gives the well-known Heisenberg relation

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2} \tag{6}$$

Beside this, Robertson inequality gives uncertainty relations between angular position and angular momentum of an object with small angular uncertainty, between two orthogonal components of the total angular momentum operator, between the number of electrons in a superconductor and the phase of its Ginzburg-Landau order parameter [2][3].

There is one uncertainty relation, which is not so obvious consequence of the Robertson relation: the time-energy uncertainty principle. Since energy has the same relation to time as momentum does to space in special relativity, it was clear to many founders of quantum mechanics (Niels Bohr among them) that the following relation holds:

$$\Delta t \Delta E \geq \frac{\hbar}{2} \tag{7}$$

Also Heisenberg [11] discusses the classically conjugate variables of time and energy and defines a time operator through the, quote, ”familiar relation”

$$[\hat{E}, \hat{t}] = -\mathrm{i}\hbar \tag{8}$$

and on the basis of this he assumes the uncertainty relation.

But it was not clear what Δt was, because the time at which a particle has given state is not an operator belonging to the particle, it is rather an external parameter describing the evolution of the system. Nevertheless, Einstein and Bohr understood the meaning of this principle. A state, which exists only for a short time, cannot have definite energy. For example, in spectroscopy, excited states have a finite lifetime and by the time-energy uncertainty principle, their energy cannot be determined – each time they decay, the energy they release is slightly different. The energy distribution of the outgoing photon has a peak at theoretical value of the state energy, but the spectrum has finite width. The same effect occurs in particle physics: it makes difficult to determine the rest masses of fast decaying particles.

II. DISCUSSION ON THE INTERPRETATION

Basically, the interpretation of Δt is not unique – it depends on the kind of experiment. It can be the lifetime of a state, but usually it is the accuracy of time measurement [4]. One false formulation of the principle is that measuring the energy of a quantum state to accuracy ΔE requires a time interval at least $\hbar/2\Delta E$:

$$\Delta t \geq \frac{\hbar}{2\Delta E} \tag{9}$$

Such variation of the principle, with Δt as a duration of the measurement, is not always satisfied.

Another and more widely-used form of the principle was given in 1945 by L.I. Mandelshtam and I.E. Tamm [6]. For an energy operator E, an observable B and a non-stationary state $|\psi\rangle$, the following formula holds:

$$\Delta E \frac{\Delta B}{\left|\frac{\mathrm{d}\langle B\rangle}{\mathrm{d}t}\right|} \geq \frac{\hbar}{2} \tag{10}$$

The second factor in the left-hand side has dimension of time, and is so-called lifetime of the state $|\psi\rangle$ with respect to the observable B. In other words, this is the time after which the expectation value of B changes appreciably.

A very rigorous discussion on the notion of time was given in [4]. The author gives a summary of the main types of time-energy uncertainty relations $\Delta t\Delta E \gtrsim \hbar$, and their range of validity depending on the interpretation of the quantities Δt and ΔE:

- A relation involving *external time* is valid if Δt is the *duration* of a perturbation or preparation process and ΔE is the uncertainty of the energy in the system.

- There is *no* limitation to the duration of an energy measurement and the disturbance or inaccuracy of the measured energy.
- There is a variety of measures of *characteristic, intrinsic times*, with ensuing *universally* valid *dynamical time-energy uncertainty relations*, ΔE being a measure of the width of the energy distribution or its fine structure. This comprises the Bohr-Wigner, Mandelshtam-Tamm, Bauer-Mello, and Hilgevoord-Uffink relations.
- *Event time observables* can be formally represented in terms of positive operator valued measures over the relevant time domain. An *observable time-energy uncertainty relation*, with a constant positive lower bound for the product of inaccuracies, is *not universally* valid but will hold in specific cases, depending on the structure of the Hamiltonian and the time domain.
- Time measurements by means of *quantum clocks* are subject to a dynamical time-energy uncertainty relation, where the time resolution of the clock is bounded by the unsharpness of its energy, $\delta t \gtrsim \hbar/\Delta E$.
- Einstein's photon box experiment constitutes a demonstration of the *complementarity* of time of passage and energy: as a consequence of the quantum clock uncertainty relation, the inaccuracy δE in the determination of the energy of the escaping photon limits the uncertainty ΔT of the opening time of the shutter. This is in accordance with the *energy measurement* uncertainty relation based on *internal clocks* discovered recently by Aharonov and Reznik [13].
- Temporal diffraction experiments provide evidence for the *objective indeterminacy* of event time uncertainties such as time of passage.

Finally we have to recall that:

- A full-fledged quantum mechanical theory of time measurements is still waiting to be developed.

In 1930, during the famous Einstein-Bohr debate on quantum mechanics, a thought experiment was designed. It was called "Einstein's box" and was supposed to violate the time-energy uncertainty relation:

Consider a box filled with light. The box has a hole in one of the walls and a shutter, which opens and quickly closes the hole, such that some of the light escapes. There is a clock, which can be set such that the moment at which the photon escapes is known. In order to measure the energy of the leaving photon, Einstein proposed weighing the box before and after the emission – the box can be suspended on a spring, there is a pointer and a scale. The difference of masses multiplied by c^2 will equal the energy of the photon.

The idea of this thought experiment was that the uncertainty of time, at which the photon escapes, can be as small as one wishes

$$\Delta t \rightarrow 0 \tag{11}$$

and the energy of photon can be measured to finite accuracy, such that

$$\Delta t \Delta E \rightarrow 0 \tag{12}$$

what is in contradiction with the time-energy uncertainty relation.

Bohr spent whole day thinking of a solution of this paradox. Finally he realized that since the box is immersed in a gravitational field, then the uncertainty in position of the box alters the ticking rate of the clock. It was ironic, because Einstein himself was the first who discovered gravity's effect on clocks.

There was another error made by Einstein: he was thinking of a photon as an localized object, with definite energy. But such an object does not exist! Since the shutter is open during vanishing time interval, the electromagnetic pulse must be very sharp (Dirac delta in the limit when interval goes to zero). And from classical electrodynamics we know, that Fourier components of such a pulse have wide spectrum of frequencies (Dirac delta is a superposition of infinite number of sinusoids with various frequencies), so the energy is not defined.

Today it is known that also Bohr's reply had some errors [1]. However, the precise evaluation of all effects shows that there always appears enough amount of uncertainty and the relation cannot be violated.

III. LAGRANGIAN FORMALISM

In an interesting paper [5] S. Kudaka and S. Matsumoto show one of possible derivations of the time-energy uncertainty relation. In the exact form, it is relation between the proper time and rest mass of an object. We will select the simplest Lagrangian, which can describe the Einstein's box and other systems of that kind (a massive object in a field) and then we will examine the Hamiltonian formalism. Notice that the weighing procedure does not require gravitational field: assume that the object has an electric charge e. Then the mass can also be determined from relation

$$e\mathcal{E} \; = \; m\,\frac{v}{t} \tag{13}$$

where $\mathcal{E}$ is external electric field switched on for a time t, and v is the velocity achieved by the object. Therefore, for generality, we assume that both gravitational ($g_{\mu\nu}$) and electromagnetic (A_μ) fields are given. Lagrangian, which is used in such cases, is the following:

$$L_0 \; = \; -mc\sqrt{-g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu} + e\,A_\mu(x)\dot{x}^\mu \tag{14}$$

where c is speed of light and m is the rest mass of the object. Here the variables are x^μ. For a clock, the proper time is a measurable physical quantity. Therefore we have to find another Lagrangian, which includes the proper time as dynamic variable (in addition to position x^μ). The other condition is that the new Lagrangian must have the same equations of motion as original Lagrangian L_0. As a candidate we consider

$$L \; = \; M\,(\dot{\tau} - \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}/c) + e\,A_\mu(x)\,\dot{x}^\mu \tag{15}$$

The dynamic variables are τ, M and x^μ. After calculating equations of motion it can be found that τ can be identified with proper time of the object, and, in order to have the same equation as one derived from L_0, M must be identified with the constant mc^2.

We assume now, for simplicity, that the fields are static in the following sense: functions $g_{\mu\nu}$ and A_μ depend only on spatial coordinates x^1, x^2, x^3, and we have $g_{i0} = 0$. It doesn't affect the generality. Then our Lagrangian can be written as

$$L \; = \; M\,(\dot{\tau} - \sqrt{f(x)^2 - g_{ij}(x)\dot{x}^i\dot{x}^j/c^2}) + c\,e\,A_0(x) + e\,A_i(x)\,\dot{x}^i \tag{16}$$

where f is defined by $g_{00} = -f^2$.

The dynamic variables are τ, M, x^i ($i = 1, 2, 3$) and the dot denotes differential with respect to t. The momentums conjugate to those variables are

$$p_\tau \; \equiv \; \frac{\partial L}{\partial \dot{\tau}} \; = \; M \tag{17}$$

$$p_M \equiv \frac{\partial L}{\partial \dot{M}} = 0 \tag{18}$$

$$\begin{aligned} p_i \equiv \frac{\partial L}{\partial \dot{x}^i} &= -M \frac{1}{2\sqrt{f^2 - g_{kj}\dot{x}^k \dot{x}^j / c^2}} \left(-\frac{g_{kj}}{c^2} (\delta_i^k \dot{x}^j + \dot{x}^k \delta_i^j) \right) + e A_k \delta_i^k = \\ &= \frac{M}{c^2} \frac{g_{ij} \dot{x}^j}{\sqrt{f^2 - g_{jk} \dot{x}^j \dot{x}^k / c^2}} + e A_i \end{aligned} \tag{19}$$

IV. HAMILTONIAN FORMALISM

We can calculate the Hamiltonian:

$$\begin{aligned} H_0 &\equiv p_\tau \dot{\tau} + p_M \dot{M} + p_i \dot{x}^i - L = \\ &= f \sqrt{M^2 + c^2 g^{ij} (p_i - e A_i)(p_j - e A_j)} - c e A_0 \end{aligned} \tag{20}$$

In our case, there exist two constraints: $\phi_1 \equiv M - p_\tau = 0$ and $\phi_2 \equiv p_M = 0$. Taking these constraints into account we have to consider the total Hamiltonian $H \equiv H_0 + u_1\phi_1 + u_2\phi_2$, where u_1 and u_2 are undetermined Lagrange's multipliers. These multipliers can be determined by consistency conditions – time-derivatives of the constraints (defined by Poisson's brackets with H) must be weakly equal to zero:

$$\{\phi_1, H\} \approx 0 \,, \quad \{\phi_2, H\} \approx 0 \tag{21}$$

This requires the Hamiltonian to be

$$H = H_0 - \frac{f M (M - p_\tau)}{\sqrt{M^2 + c^2 g^{ij} (p_i - e A_i)(p_j - e A_j)}} \tag{22}$$

Dirac's bracket, for our case, is

$$\{A, B\}_{\mathrm{D}} = \{A, B\} + \{A, \phi_1\}\{\phi_2, B\} - \{A, \phi_2\}\{\phi_1, B\} \tag{23}$$

We can easily calculate Dirac's brackets between canonical variables τ, p_τ, M, p_M, x^i, p_i:

$$\begin{aligned} \{\tau, p_\tau\}_{\mathrm{D}} &= \{\tau, p_\tau\} + \{\tau, M - p_\tau\}\{p_M, p_\tau\} - \{\tau, p_M\}\{M - p_\tau, p_\tau\} = \\ &= 1 + (-1)\cdot 0 - 0 \cdot 0 = 1 \end{aligned} \tag{24}$$

$$\begin{aligned} \{\tau, M\}_{\mathrm{D}} &= \{\tau, M\} + \{\tau, M - p_\tau\}\{p_M, M\} - \{\tau, p_M\}\{M - p_\tau, M\} = \\ &= 0 + (-1)\cdot(-1) - 0 \cdot 0 = 1 \end{aligned} \tag{25}$$

$$\{x^i, p_j\}_{\mathrm{D}} = \delta^i_j \tag{26}$$

$$\text{the others} = 0 \tag{27}$$

The following set of variables:

$$\phi_1 \equiv M - p_\tau, \quad \phi_2 \equiv p_M, \quad T \equiv \tau - p_M, \quad E \equiv p_\tau, \quad x^i, \quad p_i \quad (i = 1, 2, 3)$$

are independent variables and therefore they are canonical. In mathematical terms, conjugate variables form part of a symplectic basis. The variables T, E, x^i, p_i can be interpreted as canonical variables on the submanifold defined by the constraints $\phi_1 = 0$ and $\phi_2 = 0$.

On the submanifold we have $M = p_\tau$ and $p_M = 0$, what gives $T = \tau$ and $E = M$ ($= mc^2$). Then the Dirac's brackets take form

$$\{\tau, E\}_{\mathrm{D}} = 1 \quad , \quad \{x^i, p_j\}_{\mathrm{D}} = \delta^i_j \quad , \quad \text{the others} = 0 \tag{28}$$

It follows from the above that the rest energy $E = mc^2$ is the general momentum conjugate to the proper time. If we quantize our system by Dirac's procedure, there are corresponding operators:

$$\hat{\tau},\ \hat{E},\ \hat{x}^i,\ \hat{p}_i \ \ (i = 1,2,3) \tag{29}$$

which satisfy following commutation relations:

$$[\hat{\tau}, \hat{E}] = [\hat{x}^i, \hat{p}_i] = \mathrm{i}\hbar \tag{30}$$

We can now substitute value of the commutator $[\hat{\tau}, \hat{E}]$ to the Robertson relation (4), and it leads us to the uncertainty relation

$$c^2 \Delta m \Delta \tau \ \geq \ \frac{\hbar}{2} \tag{31}$$

This relation can be easily translated into $\Delta E \Delta t \geq \hbar/2$ when the velocity is small (the rest energy will be approximately equal to the total classical energy E, and the proper time will be approximately equal to the Newtonian time t).

V. CONCLUSIONS

It seems possible [8] to apply the above reasoning to any object (not necessarily a clock). The concept of the proper time of a particle can be always introduced by the equation

$$\mathrm{d}\tau = \sqrt{\mathrm{d}t^2 - \left(\frac{\mathrm{d}\vec{x}}{c}\right)^2} \tag{32}$$

where $(t,\ \vec{x})$ are the coordinates of the particle observed from an inertial system. Hence it is defined for every particle irrespective of whether the particle has some function as a clock or not.

The conjugate variable to the proper time is necessarily the rest energy, and therefore there exists uncertainty relation for proper time and rest mass, independent of the structure of the object. Of course, as in the case of position-momentum, this does not invalidate other forms of the time-energy uncertainty relation derived in other ways, for example a consideration of the Fourier relationship between energy and time wavepacket widths. Moreover, many different reasonings lead to the conclusion, that, besides the energy operator, there must exist a time operator, although its meaning and exact form still need clarification. For example, Pauli [14] gave a powerful and well-known argument against the existence of a time operator, based on considerations of the boundedness of the energy operator. Pauli writes ”we conclude therefore that the introduction of a time operator $\hat{t}$ must be abandoned fundamentally and that the time t in quantum mechanics has to be regarded as an ordinary real number.” Despite this, beginning with the seminal paper of Aharonov and Bohm [15], there have been numerous attempts to define an operator for time.

Another problem is that in our quantization, the operators $\hat{\tau}$, $\hat{E}$, $\hat{x}^i$ and $\hat{p}_i$ ($i = 1, 2, 3$) can be represented in the Hilbert space composed of square-integrable functions of τ, x^1, x^2 and x^3. In particular, the operator $\hat{E}$ is represented by the differential operator $-i\hbar\partial/\partial\tau$, and therefore the rest energy $\hat{E}$ cannot have any discrete spectrum. Furthermore, this Hilbert space includes some states in which the mean values of $\hat{E}$ are negative. The problems of the continuous mass spectrum and of the negative mass are inevitable in above formulation. These problems are discussed from a rather different viewpoint in [8].

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