Geometry of gauge fields

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Outline

What is gauge theory?

- Gauge theory is a field theory in which the lagrangian is invariant under a group of local transformations
- Partial derivative is redefined:

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + \mathscr{A}_\mu$$

- Transformation rules:

$$\mathscr{A}_\mu \to U(g)\mathscr{A}_\mu U(g)^{-1} + (\partial_\mu U(g))U(g)^{-1}$$

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Vector bundle

Definition

A vector bundle is a quadruple$(\mathscr{B}, \mathscr{M}, \pi, \mathscr{F})$, where $\mathscr{B}, \mathscr{M}$- smooth manifolds called total space and base respectively, $\mathscr{F}$ - n-dimensional vector space called standard fibre and $\pi : \mathscr{B} \to \mathscr{M}$ is an onto map called projection, such that following condition is satisfied:
Let $\{\mathscr{U}_\alpha\}$ be the open covering of$\mathscr{M}$. Then, for everyα there exists a diffeomorphism:

$$t_\alpha : \pi^{-1}(\mathscr{U}_\alpha) \longrightarrow \mathscr{U}_\alpha \times \mathscr{F}$$

such that its restriction $t_{\alpha,p} \equiv t_\alpha|_p : \mathscr{B}_p \equiv \pi^{-1}(p) \longrightarrow \{p\} \times \mathscr{F}$ is a linear isomorphism. This diffeomorphism (together with a set $\mathscr{U}_\alpha$) is called a local trivialization of a bundle.

Vector bundle

Example

A cylinder is an example of a vector bundle. In this case whole cylinder is identified with the total space, S^1 is a base manifold and $\mathscr{R}^1$ is a standard fibre. An arbitrary local trivialization is of the form $\mathscr{U}_\alpha \times \mathscr{R}^1$. In this special case one can choose even $\mathscr{U}_\alpha = \mathscr{S}^1$, what makes a cylinder being a trivial vector bundle.

Transition function

$$g_{\alpha\beta}(p) = t_\alpha(p) \circ t_\beta(p)^{-1}$$

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Section of a vector bundle

Definition

A function $s : \mathscr{M} \longrightarrow \mathscr{B}$ such that for every $p \in \mathscr{M}$, $s(p) \in \mathscr{B}_p$ (equivalently, $\Pi \circ s = Id$) is called a section of a vector bundle. The set of all sections of a bundle $\mathscr{B}$ will be denoted as $\Gamma(\mathscr{B})$.

Basis of sections

We say that $e_1, e_2 \ldots e_n \in \Gamma(\mathscr{B})$ form a basis of sections, if every $s \in \Gamma(\mathscr{B})$ can be written as $s = s^i e_i$, where s^i are appropriate functions from $C^\infty(\mathscr{M})$.

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Gauge transformation

Definition

We say, that $T(p) \in End(\mathscr{B}_p)$ lives in $\mathscr{G}$, if it is of the form $v \to gv$ for some $g \in \mathscr{G}$ and $v \in \mathscr{B}_p$. If $T(p)$ lives in $\mathscr{G}$ for every p, then we call T a gauge transformation.

Definition

Connection on a vector bundle is a map:

$$\mathscr{D} : \Gamma(\mathscr{B}) \longrightarrow \Gamma(T^*\mathscr{M} \otimes \mathscr{B})$$

which satisfies the following conditions:

- For any $s_1, s_2 \in \Gamma(\mathscr{B})$:

$$\mathscr{D}(s_1 + s_2) = \mathscr{D}(s_1) + \mathscr{D}(s_2)$$

- For $s \in \Gamma(\mathscr{B})$ and $f \in C^\infty(\mathscr{M})$:

$$\mathscr{D}(fs) = df \otimes s + f\mathscr{D}s$$

Transformation rules

$$\omega' = dAA^{-1} + A\omega A^{-1}$$

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Covariant derivative

Definition

Let X be a smooth vector field on a base manifold $\mathscr{M}$. The covariant derivative of a section s along X is a map:

$$D_X : \Gamma(B) \longrightarrow \Gamma(B)$$

$$D_X s \equiv < X, Ds >$$

Coordinate representation

$$(D_\mu s)^\alpha = \partial_\mu s^\alpha + s^\beta \omega^\alpha_{\beta\mu}$$

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For Further Reading I

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