Geometry of gauge fields

Michal Kosztolowicz

June 15, 2010

Abstract

The aim of this article is to give a mathematical formulation of Yang-Mills gauge theory. Formalism of vector bundles and interpretation of a field as a section of a vector bundle will be introduced. Using this language notion of gauge transformation will be given and finally a connection on a vector bundle will be defined, an object to be recognised as a gauge potential.

1 Introduction

A gauge theory is a field theory in which the lagrangian is invariant under a group of local transformations. Starting with the lagrangian which invariance is only global, we can achieve a local invariance at the expense of introducing a gauge field $\mathcal{A}_\mu$. It is used in a redefinition of a partial derivative inserted in the lagrangian:

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + \mathcal{A}_\mu$$

Transformation rules of this new field are given as follows:

$$\mathcal{A}'_\mu = U(g)\mathcal{A}_\mu U(g)^{-1} + (\partial_\mu U(g))U(g)^{-1}$$

where $U(g)$ is a local gauge transformation.

The rest of this paper will be dedicated to introducing a mathematical object that has above mentioned properties.

2 Vector bundles

Definition 1 *A vector bundle is a quadruple $(\mathcal{B}, \mathcal{M}, \pi, \mathcal{F})$, where $\mathcal{B}, \mathcal{M}$- smooth manifolds called total space and base respectively, $\mathcal{F}$ - n dimensional vector space called standard fibre and $\pi : \mathcal{B} \to \mathcal{M}$ is an onto map called projection, such that following condition is satisfied:*

Let $\{\mathcal{U}_\alpha\}$ be the open covering of $\mathcal{M}$. Then, for every α there exists a diffeomorphism

$$t_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \longrightarrow \mathcal{U}_\alpha \times \mathcal{F}$$

such that its restriction $t_{\alpha,p} \equiv t_\alpha|_p : \mathcal{B}_p \equiv \pi^{-1}(p) \longrightarrow \{p\} \times \mathcal{F}$ is a linear isomorphism. This diffeomorphism (together with a set $\mathcal{U}_\alpha$) is called a local trivialization of a bundle.

It is worth emphasizing that it is enough there exists at least one open covering satisfying condition from the definition to be able to talk about a vector bundle. This covering dooes not need to be an atlas from differentiable structure on $\mathcal{M}$. The notion of a fibre, defined as $\mathcal{B}_p = \pi^{-1}(p)$, is a subset of total space. In arbitrary local trivialization, such that $p \in \mathcal{U}_\alpha$, this fibre is isomorphic to the standard fibre $\mathcal{F}$.

The idea of a vector bundle appears naturally in mathematics, what is easily seen in a following

Example 2 *A cylinder is an example of a vector bundle. In this case whole cylinder is identified with the total space, $\mathcal{S}^\infty$ is a base manifold and $\mathcal{R}^1$ is a standard fibre. An arbitrary local trivialization is of the form $\mathcal{U}_\alpha \times \mathcal{R}^1$. In this special case one can choose even $\mathcal{U}_\alpha = \mathcal{S}^\infty$, what makes a cylinder being a trivial vector bundle.*

It is instructive to consider a different vector bundle that in local trivialization looks like $\mathcal{U}_\alpha \times \mathcal{R}^1$, namely a Mobius strip. It is obvious that although locally identical globaly it is a different object than a cylinder.

Suppose now that there are two trivializations (t_α, U_α) and (t_β, U_β) of the same vector bundle, such that $U_\alpha \cup U_\beta \neq \emptyset$. It makes sense to introduce a transition function, defined as $g_{\alpha\beta}(p) = t_\alpha(p) \circ t_\beta(p)^{-1}$ for $p \in U_\alpha \cup U_\beta$. This mapping transforms vectors from a fibre while changing a local trivialization. In gauge theory one demands that it is not an arbitrary isomorphism, but rather an isomorphism from a certain group $\mathcal{G}$, called gauge group.

The notion of a vector bundle can be generalized using construction that can be made of a single vector space. Consider for example a dual space $\mathcal{F}^*$. Then, there is a well defined notion of a dual vector bundle $\mathcal{B}^*$. One can think of it simply as a vector bundle which fibre is a dual space $\mathcal{F}^*$. In a similar way one can construct tensor bundle $\mathcal{B} \otimes \mathcal{B}^*$ etc.

3 Section of a vector bundle

Definition 3 *A function $s : \mathcal{M} \longrightarrow \mathcal{B}$ such that for every $p \in \mathcal{M}$ $s(p) \in \mathcal{B}_p$ (equivalently, $\Pi \circ s = id$) is called a section of a vector bundle. The set of all sections of a bundle $\mathcal{B}$ will be denoted as* $\Gamma(\mathcal{B})$

That means that a section is a function that for every point p from the base manifold picks out a single vector from a fibre $\pi^{-1}(p)$. Shortly speaking, section of a vector bundle is a vector field over a base manifold.

There always exists a global section of a vector bundle (that is defined on the whole $\mathcal{M}$, not only on a certain open subset). For example, one can consider the zero section - it maps every point into zero vector and is independent of local trivialization, since zero vector is always mapped into zero vector by an isomorphism.

In a natural way there can be an addition and a multiplication by a function from $C^\infty(\mathcal{M})$ introduced in the set $\Gamma(\mathcal{B})$. We define:

$$(s + s')(p) \equiv s(p) + s'(p)$$

$$(fs)(p) \equiv f(p)s(p)$$

for arbitrary $s, s' \in \Gamma(\mathcal{B})$ and $f \in C^\infty(M)$. Using above it makes sense to talk about a linear dependence of sections:

Definition 4 *We say that $e_1, e_2 \ldots e_n \in \Gamma(\mathcal{B})$ form a basis of sections, if every $s \in \Gamma(\mathcal{B})$ can be written as*

$$s = s^i e_i$$

where s^i are appropriate functions from $C^\infty(\mathcal{M})$.

A basis of sections can be only defined for trivial bundles. That means that for an arbitrary vector bundle we can have a basis only localy (for local trivializations).

4 Gauge transformation

Using above introduced formalism notion of a gauge transformation can be defined. For this purpose one needs a section of an endomorphism bundle which is the same as just mentioned tensor bundle $\mathcal{B}^* \otimes \mathcal{B}$, due to the isomorphism $\mathcal{V}^* \otimes \mathcal{V} \cong End(\mathcal{V})$, where $\mathcal{V}$ is an arbitrary, finite dimensional vector space. Let $T \in \Gamma(\mathcal{B}^* \otimes \mathcal{B})$. It can be shown that $T(p) \in \mathcal{B}_p^* \otimes \mathcal{B}_p \cong End(\mathcal{B}_p)$, so it is a linear map acting on vectors from $\mathcal{B}_p$. Thus, given a section $s \in \Gamma(\mathcal{B})$ T defines a new section $T(s) \in \Gamma(B)$ as follows:

$$T(s)(p) = T(p)s(p)$$

After this introduction a definition can be given:

Definition 5 *We say, that $T(p) \in End(\mathcal{B}_p)$ lives in $\mathcal{G}$, if it is of the form $v \to gv$ for some $g \in \mathcal{G}$ and $v \in \mathcal{B}_p$. If $T(p)$ lives in $\mathcal{G}$ for every p, then we call T a gauge transformation.*

Since a vector field is defined as a section of a vector bundle and a field equation is a differential equation in general, one needs to know how to differentiate sections. It is not a straightforward problem, since a first guess to define it as for usual functions would require an operation of addition of vectors form different fibres - an operation that is not cannonicaly given. For this reason a notion of connection is provided.

5 Connection on a vector bundle

5.1 Definition and transformation rules

Definition 6 *Connection on a vector bundle is a map*

$$\mathcal{D} : \Gamma(\mathcal{B}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes \mathcal{B})$$

which satisfies the following conditions:

1. *For any* $s_1, s_2 \in \Gamma(\mathcal{B})$

$$\mathcal{D}(s_1 + s_2) = \mathcal{D}(s_1) + \mathcal{D}(s_2)$$

2. *For* $s \in \Gamma(\mathcal{B})$ *and* $f \in C^\infty(\mathcal{M})$

$$\mathcal{D}(fs) = df \otimes s + f\mathcal{D}s$$

Suppose now that $(\mathcal{U}_\beta, t_\beta)$ is a local trivialization of $T^*\mathcal{M} \otimes \mathcal{B}$ and u^i are local coordinates on $\mathcal{U}$. Then $du^i \otimes e_\alpha$ forms a basis of $T_p^*\mathcal{M} \otimes \mathcal{B}_p$ for every $p \in \mathcal{U}$, where e_α is a basis of sections of vector bundle $\mathcal{B}$ on $\mathcal{U}$. Since $\mathcal{D}e_\alpha$ is a local section on $\mathcal{U}$ of $T^*\mathcal{M} \otimes \mathcal{B}$, it can be written locally as

$$\mathcal{D}e_\alpha = \omega^\beta_{\alpha i} du^i \otimes e_\beta$$

where $\omega^\beta_{\alpha i}$ are smooth functions on $\mathcal{U}$.

Using notation

$$\omega^\beta_\alpha = \omega^\beta_{\alpha i} du^i$$

one obtains

$$\mathcal{D}e_\alpha = \omega^\beta_\alpha \otimes e_\beta$$

It is convienient to introduce a matrix notation, that is

$$S = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

$$\omega = \begin{pmatrix} \omega^1_1 & \cdots & \omega^n_1 \\ \vdots & \ddots & \\ \omega^1_n & \cdots & \omega^n_n \end{pmatrix}$$

Then we end up with a simple realtion

$$\mathcal{D}S = \omega \otimes S \tag{1}$$

where ω is called the connection matrix. If a basis of sections is chosen, connection is locally given by a matrix ω. Change of a basis of sections corresponds to a transformation of physical fields. We want to figure out transformation properties of the connection matrix. For this reason consider change of a basis of sections given by a matrix A

$$S' = AS$$

Putting it into (1) one finds

$$\mathcal{D}S' = \mathcal{D}(AS) = dA \otimes S + A\mathcal{D}S = dA \otimes S + A(\omega \otimes S) = (dA + A\omega) \otimes S =$$
$$= (dAA^{-1} + A\omega A^{-1}) \otimes S' = \omega' \otimes S'$$

An improtant formula for a transformation of connection is then derived:

$$\omega' = dAA^{-1} + A\omega A^{-1}$$

One immediately notices that this is precisely a transformation rule of a potential of a gauge field. That is why those objects are identified.

5.2 Covariant derivative

Having a notion of a connection introduced, we can give a following

Definition 7 *Let X be a smooth vector field on a base manifold M. The covariant derivative of a section s along X is a map $D_X : \Gamma(B) \longrightarrow \Gamma(B)$*

$$D_X s \equiv < X, Ds >$$

If $X = \partial_\mu$, we denote $D_{\partial_\mu} \equiv D_\mu$. We can rewrite $D_\mu s$ in terms of basis sections e_α of $\Gamma(B)$ as follows:

$$D_\mu s =< \partial_\mu, Ds >=< \partial_\mu, D(s^\alpha e_\alpha) >=< \partial_\mu, ds^\alpha \otimes e_\alpha + s^\alpha De_\alpha >=$$
$$=< \partial_\mu, ds^\alpha \otimes e_\alpha > + < \partial_\mu, s^\alpha \omega^\beta_\alpha \otimes e_\beta >= ds^\alpha(\partial_\mu)e_\alpha + s^\alpha \omega^\beta_\alpha(\partial_\mu)e_\beta =$$
$$= \partial_\mu s^\alpha e_\alpha + s^\alpha \omega^\beta_{\alpha\mu} e_\beta = (\partial_\mu s^\alpha + s^\beta \omega^\alpha_{\beta\,\mu})e_\alpha$$

As a result, we arrive at a desired formula for a coordinate representation of the covariant derivative:

$$(D_\mu s)^\alpha = \partial_\mu s^\alpha + s^\beta \omega^\alpha_{\beta\,\mu}$$

References

[1] J. Baez, J.P. Muniain: *Gauge fields, knots and gravity*, World Scientific Publishing Co. Pte. Ltd., Singapore 1994.

[2] L. Fatibene, M. Francaviglia: *Natural and gauge natural formalism for classical field theories. A geometric perspective including spinors and gauge theories*, Kluwer Academic Publishers, Dordrecht/Boston/London 2003.

[3] S.S. Chern, W.H.Chen, K.S. Lam: *Lectures on differential geometry*, World Scientific Publishing Co. Pte. Ltd., Singapore 2000.

[4] Ch. N. Yang: *Selected papers (1945-1980) with commentary*, World Scientific Publishing Co. Pte. Ltd., Singapore 2005.