General Reference Frame

Aneta Wojnar

June 17, 2010

Abstract

The aim of the article is to show how the notion of reference frame was evolving from ancient times to nowadays. The definitions of Aristotle's spacetime, Galilei spacetime, spacetime in special relativity will be given. Finally, general reference frame will be defined in general relativity formalism.

1 Aristotle's spacetime

Definition 1 ***Elementary event*** *is a physical phenomenon in a small area in space lasting very shortly. Features of this event are not important.*

Definition 2 *Set of all elementary events is called* **spacetime**. *By a* ***world line*** *of a material point one will understand a history of this point in spacetime.*

Till Galilei and Newton's time it was believed that space and time are absolute quantities; it means that any two events have unambiguously defined distance between them in space and unambiguously defined distance in time. One supposed that among all possible reference frames there exists a frame which is at rest and in this frame one should measure the distances. The assumption about existence of spatialy distinguished reference frame and absolute time means that the spacetime E one can present as a Cartesian product:

Definition 3 ***Aristotle's spacetime*** *is a set:*

$$E = T \times S = \{(t, x); \quad t \in T, \quad x \in S\} \tag{1}$$

where $T = E^1$ *is a time axis and* $S = E^3$ *is a three-dimensional position space.* E^n *means n-dimensional Euclidean space.*

It can be seen that in Aristotle's physics one can consider an absolute distance between two events is space S even when the events are not simultaneous. What is more, Aristotle assumed existence of a notion of absolute rest, what means that there exists a distinguished reference frame which is always at rest. Refering to the mathematical model (1), straight lines in $E = T \times S$ consisting of points which relative distances vanish, it means sets of events: $\{(t, x); x = const\}$, represent world lines of objects at absolute rest.

2 Galilei spacetime

In the seventeenth century one rejected the assumption about existence of reference frame distinguished by nature. In consequence one gave up the idea of Aristotle's spacetime model.

The difference between Aristotle's and Galilei spacetime is that in the latter spatial distance between two events is defined only when those events are simultaneus. The space became relative, so the distance between two events, which do not take place in the same time, depends on a choice of reference frame, but the time is still absolute. It means that the difference of times between two events does not depend on reference frame choice. However, there was distinguished preferred class of frames in which all measurements of forces acting on the body give the same results. They are called **inertial frames**, so the following statement is true:

A particle not subject to any force moves with constant velocity.

It is obvious that this statement is false if the particle is looked at from an accelerated frame, which are called non-inertial frames.

For that reason, inertial frames are consequently very special and are used as the basic frames. They are attributed with Cartesian coordinates $\vec{x} = (x_1, x_2, x_3)$ and the variable t is used to indicate time.

If there are two frames, θ and θ' that coincide at $t = 0$ and later θ' is moving with constant velocity $\vec{u}$, the value of the position, seen from θ', will be:

$$\vec{x'} = \vec{x} - \vec{u} \cdot t \tag{2}$$

where $t = t'$. This simple law is the same for the composition of velocity, obviously.

If a third frame θ" displaces itself with constant velocity with respect to θ', it will move with constant velocity with respect to θ and will be inertial too. For that reason, all inertial frames are equivalent, that is *the relation of being inertial is reflexive, symmetric and transitive.*

One can sum up:

There exist preferential reference systems, called by definition ***inertial frames****, in which*:

- *The laws of nature are the same in all of inertial frames (Galilean relativity)*
- *There exist preferential reference systems for which, given initial inertial frame, all the other inertial systems are in uniform rectilinear motion with respect to it (inertia).*

3 Einstein's and Minkowski's spacetime

At the end of nineteenth century there was a problem with the unobservable ether which for the light would be the same as air is for sound. A lot of experiments shown that the speed of light is the same in all inertial frames at all times and in all directions, independently of the motion of the source and the observer. So the speed of light does not obey the classical velocity law.

In 1905 Einstein postulated a generalization of Galilean Principle of Relativity which is called Einstein's Principle of Relativity:

1. *Physical laws have the same form in all inertial frames.*
2. *The speed of light is finite and equal in all inertial frames.*

Although one can see that the first postulate has the form similar to Galilean Principle of Relativity (PR), there is an essential difference. It should be remembered that Galilean PR is realized in classical mechanics, in terms of Galilean transformations and the classical velocity addition law, which does not hold for light signals. The realization of Einstein's PR is given in terms of Lorentz transformation, so there is a new connection between two inertial frames and a new law for the addition of velocities. When one of the systems is moving with constant velocity $\vec{u}$ along $\vec{x}$ axis, the transformation law will be:

$$\vec{x}' = \frac{\vec{x} - \vec{u} \cdot t}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad t' = \frac{t - \frac{u \cdot x}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad y' = y \qquad z' = z \tag{3}$$

$$\Downarrow$$

$$\vec{u}_x' = \frac{\vec{u}_x - \vec{u}}{\sqrt{1 - \frac{\vec{u}_x \vec{v}}{c^2}}} \tag{4}$$

4 General Relativity

In Newtonian-Galilean mechanics and in special relativity, the law of inertia distinguishes a special class of equivalent frames of reference (inertial systems). Due to the universality of gravitation, only the free fall of electrically neutral test bodies can be regarded as particularly distinguished motion in the presence of gravitational fields. Such bodies experience, however, relative acceleration. Hence the law of inertia is not longer valid and the concept of an inertial system cannot be defined globally. This implies **the principle of equivalence**:
At every point in an arbitrary gravitational field we can choose a locally inertial frame in which the laws of physics take the same form as in special relativity.

In general relativity, the principle of inertia is replaced with the principle of geodesic motion, whereby objects move in a way dictated by the curvature of spacetime. As a consequence of this curvature two objects moving at a particular rate of velocity with respect to each other will not continue to do so. It means that inertial frames of reference does not exist globally as they do in Newtonian mechanics and special relativity. However, the general theory reduces to the special theory over sufficiently small regions of spacetime, where curvature effects become less important and the earlier inertial frames arguments can come back into play.

While the notion of an inertial reference frame in a Minkowski space is easy to define, the notion of general reference frame in a general spacetime is not. Reference frame is understood as a coordinate system, which mathematically is a local chart χ defined in some open domain U of spacetime manifold V.

If one changes the spatial coordinates in a way that does not depend on the time coordinate:

$$x'^0 = f((x^\mu)), \quad x'^j = f^j((x^k)); \; (j, k = 1, 2, 3), \tag{5}$$

then any element of matter which has time-independent spatial coordinates in that system, will obviously remain „at rest“ in the new system of coordinates.

Let us consider a case in which only spatial coordinates change, it means:

$$x'^0 = x^0, \quad x'^j = f^j((x^k)) \tag{6}$$

This is due to the fact, that the Hamiltonian operator H in a curved spacetime is invariant only under purely spatial changes. This fact applies to any wave equation of relativistic quantum mechanics that its wave function ψ transforms either as a scalar or as a four-vector. In other words, the Hamiltonian operator depends on the reference frame and on the choice of the time coordinate. Reference frame will be essentially an equivalence class of charts modulo the relation (6).

A formal definition of general reference frame, which will be introduced below, allows to associate rigorously with each reference frame the unique „space” M, which is a three-dimentional differentiable manifold.

Equation (6) introduces a relation between two coordinate systems or charts χ and χ' on the spacetime V. That relation involves the transition map:

$$F \equiv \chi' \circ \chi^{-1} \quad with \quad \mathbf{X}' \equiv F(\mathbf{X}) \equiv (x'^\mu) \equiv F^\mu(\mathbf{X}); \quad \mathbf{X} \equiv (\mathbf{x}^\mu) \tag{7}$$

It should be noticed that the domain of the map F is the intersection of two charts $\chi(Dom(\chi) \cap Dom(\chi'))$. Therefore, to define an equivalence relation between charts based on (6), one have to limit oneself to charts whose domains all contain some open subset $U \cap V$. Now it can be given a definition of a general reference frame F:

Definition 4 (and theorem) *Let V be a differentiable manifold of dimension $N+1$ $(N \geq 1)$ and let $\mathcal{A}$ be the atlas that defines the manifold structure on V. For any open set $U \subset V$, define the subset $\mathcal{A}_U \equiv \{\chi \in \mathcal{A};\ Dom(\chi) \supset U\}$ of the atlas $\mathcal{A}$. Let the open set U be such that $\mathcal{A}_U$ is non-empty and, for any two charts χ, $\chi' \in \mathcal{A}_U$, set:*

$$\chi \mathcal{R}_U \chi' \quad iff \quad [\forall \boldsymbol{X} \in \chi(U),\ F^0 = x^0 \ and\ \frac{\partial F^j}{\partial x^0}(\boldsymbol{X}) = 0 \quad (j = 1, ..., N)], \tag{8}$$

*where $F \equiv \chi' \circ \chi^{-1}$ is the transition map, which domain $\chi(Dom(\chi) \cap Dom(\chi'))$ contains $\chi(U)$. Then $\mathcal{R}_U$ is an equivalence relation on $\mathcal{A}_U$. The equivalence classes for this relation are called **reference frames** (over the domain U).*

Proof. To prove the theorem, one has to show that the relation $\mathcal{R}_U$ is reflexive, symmetric and transitive.

- By definition, $U \subset Dom(\chi)$, whence $\chi(U) \subset \chi(Dom(\chi))$, for all $\chi \in \mathcal{A}_U$. It follows that $\forall \chi \in \mathcal{A}_U$, $\chi \mathcal{R}_U \chi$, since $F = \chi \circ \chi^{-1} = \mathrm{Id}_{\chi(Dom(\chi))}$ in that case. That is, the relation $\mathcal{R}_U$ is reflexive.

- If $\chi\mathcal{R}_U\chi'$, let $\mathbf{X}' \in \chi'(U)$, so that the transition map $G \equiv \chi \circ \chi'^{-1} = F^{-1}$ (because $F \equiv \chi' \circ \chi^{-1}$) is defined in a neighborhood of $\mathbf{X}'$. It is obvious that $\mathbf{X} \equiv G(\mathbf{X}') = \chi \circ \chi'^{\mathbf{-1}}(\mathbf{X}') = \chi(\chi'^{-1}(\mathbf{X}')) = \mathbf{X} \in \chi(U),)$ and in a similar way: $\mathbf{X}' \equiv F(\mathbf{X})$. In particular, $x'^0 = F^0(\mathbf{X}) = \mathbf{x^0} \equiv G^{\mathbf{0}}(\mathbf{X}')$. Furthemore, keeping in mind that $G(\mathbf{X}') = G(F(\mathbf{X}))$, one has:

$$0 = \frac{\partial G^j}{\partial x^0} = \frac{\partial G^j}{\partial x'^\nu}\frac{\partial F^\nu}{\partial x^0} = \frac{\partial G^j}{\partial x'^0}\frac{\partial F^0}{\partial x^0} + \frac{\partial G^j}{\partial x'^i}\frac{\partial F^i}{\partial x^0} = \frac{\partial G^j}{\partial x'^0}, \tag{9}$$

hence $\chi'\mathcal{R}_U\chi$ and the relation $\mathcal{R}_U$ is symmetric.

- If $\chi\mathcal{R}_U\chi'$ and $\chi'\mathcal{R}_U\chi''$, for $F = \chi' \circ \chi^{-1}$, one has:

$$\forall \mathbf{X} \in \chi(U), \quad F'^0(\mathbf{X}') = x^0 \textit{ and } \frac{\partial F'^j}{\partial x'^0}(\mathbf{X}) = 0 \tag{10}$$

as well as (setting $F' \equiv \chi'' \circ \chi'^{-1}$):

$$\forall \mathbf{X}' \in \chi'(U), \quad F'^0(\mathbf{X}') = x'^0 \textit{ and } \frac{\partial F'^j}{\partial x'^0}(\mathbf{X}') = 0, \tag{11}$$

and therefore, with $F'' = \chi'' \circ \chi^{-1} = F' \circ F$,

$$\forall \mathbf{X} \in \chi(U), \qquad \mathbf{X}' \equiv F(\mathbf{X}) \in \chi'(U) \text{ and}$$

$$F''^0(\mathbf{X}) = (F' \circ F)^0(\mathbf{X}) = F'^0(F(\mathbf{X})) = F'^0(x^0, (x'^j)) = x^0. \tag{12}$$

Moreover, one can write
$\forall \mathbf{X} \in \chi(U)$,

$$\frac{\partial F''^j}{\partial x^0} = \frac{\partial F'^j}{\partial x'^0}\frac{\partial F^0}{\partial x^0} + \frac{\partial F'^j}{\partial x'^i}\frac{\partial F^i}{\partial x^0} = 0 \cdot \frac{\partial F^0}{\partial x^0} + \frac{\partial F'^j}{\partial x'^i} \cdot 0 = 0. \tag{13}$$

Equations (12) and (13) show that $\chi\mathcal{R}_U\chi''$, so the relation $\mathcal{R}_U$ is also transitive.

■

References

[1] Mayeul Arminjon and Frank Reifler: *General reference frames and their associated space manifold.* http://arxiv.org/abs/1003.3521.

[2] R.S. Ingarden, A. Jamiołkowski: *Mechanika klasyczna*, PWN, 1980.

[3] R. Aldrovandi and J.G. Pereira: *Notes for a Course on Classical Fields*, 2004.

[4] M. Blagojevic: *Gravitation and Gauge Symmetries*, IOP, 2002.

[5] N. Straumann: *General relativity and relativistic astrophysics*, Springer Verlag, 1984.

[6] L. Landau, E. Lifshitz: *The Classical Theory of Fields*, Oxford, 1980.