

1 Ensembles and Partition Function

In equilibrium statistical mechanics, one normally encounters three types of ensemble. The *microcanonical ensemble* is used to describe an isolated system which has a fixed energy E , a fixed particle number N , and a fixed volume V . The *canonical ensemble* is used to describe a system in contact with a heat reservoir at temperature T . The system can freely exchange energy with the reservoir. Thus T , N , and V are fixed. In the *grand canonical ensemble*, the system can exchange particles as well as energy with a reservoir. In this ensemble, T , V , and the chemical potential μ are fixed variables.

In the latter two ensembles $T^{-1} = \beta$ may be thought of as a Lagrange multiplier which determines the mean energy of the system. Similarly, μ may be thought of as a Lagrange multiplier which determines the mean number of particles in the system. In a relativistic quantum system where particles can be created and destroyed, it is most straightforward to compute observables in the grand canonical ensemble. So we therefore use this ensemble. This is without loss of generality since one can pass over to either of the other ensembles by performing a Laplace transform on the variable μ and/or the variable β .

Consider a system described by a Hamiltonian \hat{H} and a set of conserved number operators \hat{N}_i . In relativistic QED, for example, the number of electrons minus the number of positrons is a conserved quantity, not the electron number or the positron number separately. These number operators must be Hermitian and must commute with H as well as with each other. Also, the number operators must be extensive in order that the usual macroscopic thermodynamic limit can be taken. The statistical density matrix is

$$\hat{\rho} = \exp \left[-\beta(\hat{H} - \mu_i \hat{N}_i) \right] , \quad (1)$$

where a summation over i is implied. The ensemble average of an operator \hat{A} is

$$A = \frac{\text{Tr} \hat{\rho} \hat{A}}{\text{Tr} \hat{\rho}} . \quad (2)$$

The grand canonical partition function is

$$Z = \text{Tr} \hat{\rho} . \quad (3)$$

The function $Z = Z(T, V, \mu_1, \mu_2, \dots)$ is the single most important function in thermodynamics. From it all other standard thermodynamic properties may be determined. For example the equations of state for, e.g., the pressure P , the particle number N , the entropy S , and the energy E are, in the infinite volume limit,

$$P = T \frac{\partial \ln Z}{\partial V} , \quad (4)$$

$$N_i = T \frac{\partial \ln Z}{\partial \mu_i} , \quad (5)$$

$$P = \frac{\partial(T \ln Z)}{\partial T}, \quad (6)$$

$$E = -PV + TS + \mu_i N_i. \quad (7)$$

Note that the notion of ensembles as introduced here for the situation in thermodynamical equilibrium can be extended to the nonequilibrium situation, where a generalized Gibbs ensemble can be introduced for systems which are in a nonequilibrium situation that can be characterized by further observables such as currents or reaction variables. These additional observables shall be accounted for by an enlarged set of Lagrange multipliers thus arriving at a statistical operator of the nonequilibrium state, also called *relevant statistical operator* within the Zubarev formalism.

1.1 Partition function in Quantum Statistics and Quantum Field Theory

In order to calculate the partition using methods of quantum field theory, we recall that in quantum statistics

$$Z = \text{Tr} e^{-\beta(\hat{H} - \mu_i \hat{N}_i)} = \int d\phi_a \langle \phi_a | e^{-\beta(\hat{H} - \mu_i \hat{N}_i)} | \phi_a \rangle, \quad (8)$$

where the sum runs over all (eigen-)states. This has an appearance very similar to the transition amplitude (time evolution operator) in Quantum Field Theory when one switches to an *imaginary time* variable $\tau = i t$ and limit the integration over τ to the region between 0 and β . The trace operation means that we have to integrate over all fields ϕ_a . Finally, if the system admits some conserved charge, then we must make the replacement

$$\mathcal{H}(\pi, \phi) \rightarrow \mathcal{K}(\pi, \phi) = \mathcal{H}(\pi, \phi) - \mu \mathcal{N}(\pi, \phi), \quad (9)$$

where $\mathcal{N}(\pi, \phi)$ is the conserved charge density.

In fact, we can express the partition function Z as a functional integral over fields and their conjugate momenta. This fundamental formula reads

$$Z = \int \mathcal{D}\pi \int_{\text{periodic}} \mathcal{D}\phi \exp \left\{ \int_0^\beta \int d^3x \left(i \pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}(\pi, \phi) + \mu \mathcal{N}(\pi, \phi) \right) \right\}. \quad (10)$$

The term “periodic” means that the integration over the field is constrained so that $\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$. This is a consequence of the trace operation, setting $\phi_a(\vec{x}) = \phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$. There is no restriction on the π integration. The generalization of (10) to an arbitrary number of fields and conserved charges is obvious.

In the following subsection we show the equivalence of the expressions (10) and (8) for the partition function. The key lesson is that the quantization in the Path integral representation is provided by the integration over all alternative *classical field* configurations (under given constraints) whereas in the statistical operator representation the notion of *field operators* has to be introduced.

1.2 Equivalence of Path Integral and Statistical Operator representation for the Partition function

Be $\hat{\phi}(\vec{x}, 0)$ a field operator in the Schrödinger picture at time $t = 0$ and $\hat{\pi}(\vec{x}, 0)$ the corresponding canonically conjugated field momentum operator. For eigenstates $|\phi\rangle$ of the field holds the eigenvalue equation

$$\hat{\phi}(\vec{x}, 0) |\phi\rangle = \phi(\vec{x}) |\phi\rangle , \quad (11)$$

where $\phi(\vec{x})$ is the “eigenvalue” corresponding to the field operator. For the eigenstates of the fields completeness and orthonormality shall hold

$$\int d\phi(\vec{x}) |\phi\rangle \langle \phi| = 1 \quad (12)$$

$$\langle \phi_a | \phi_b \rangle = \delta [\phi_a(\vec{x}) - \phi_b(\vec{x})] . \quad (13)$$

For the field momentum operator and its eigenstates $|\pi\rangle$ holds analogously

$$\hat{\pi}(\vec{x}, 0) |\pi\rangle = \pi(\vec{x}) |\pi\rangle \quad (14)$$

$$\int \frac{d\pi(\vec{x})}{2\pi} |\pi\rangle \langle \pi| = 1 \quad (15)$$

$$\langle \pi_a | \pi_b \rangle = \delta [\pi_a(\vec{x}) - \pi_b(\vec{x})] . \quad (16)$$

The transition amplitude between coordinate and momentum eigenstates in quantum mechanics is ($\hbar = 1$)

$$\langle x | p \rangle = e^{ipx} . \quad (17)$$

The generalization to the quantum field theory case follows by going over to an infinite number of degrees of freedom $\sum p_i x_i \rightarrow \int d^3x \pi(\vec{x}) \phi(\vec{x})$ thus arriving at

$$\langle \phi | \pi \rangle = \exp \left[i \int d^3x \pi(\vec{x}) \phi(\vec{x}) \right] . \quad (18)$$

For a dynamical description of the system we require the Hamiltonian operator

$$\hat{H} = \int d^3x \mathcal{H}(\hat{\pi}, \hat{\phi}) . \quad (19)$$

Consider the state $|\phi_a\rangle$ at $t = 0$. At a later time t_f this state has evolved to $e^{-\hat{H}t_f} |\phi_a\rangle$. The transition amplitude of the state $|\phi_a\rangle$ to the state $|\phi_b\rangle$ at time t_f is therefore given by $\langle \phi_b | e^{-\hat{H}t_f} | \phi_a \rangle$.

In order to express the quantum statistical partition function, we are interested in the case that the system returns at $t = t_f$ to the initial state at $t = 0$. The sign of the state is not an observable and is left undetermined at this stage.

$$e^{-i\hat{H}t_f} | \phi_a \rangle \rightarrow \pm | \phi_a \rangle \quad (20)$$

In order to evaluate the transition amplitude the time interval $(0, t_f)$ is decomposed into equidistant parts of the length $\Delta t = t_f/N$. At each time step we introduce a complete set of field and field-momentum states

$$\begin{aligned} \langle \phi_a | e^{-iHt_f} | \phi_a \rangle &= \lim_{N \rightarrow \infty} \int \left(\prod_{i=1}^N \frac{d\pi_i d\phi_i}{2\pi} \right) \\ &\times \langle \phi_a | \pi_N \rangle \langle \pi_N | e^{-iH\Delta t} | \phi_N \rangle \langle \phi_N | \pi_{N-1} \rangle \\ &\times \langle \pi_{N-1} | e^{-iH\Delta t} | \phi_{N-1} \rangle \times \dots \\ &\times \langle \phi_2 | \pi_1 \rangle \langle \pi_1 | e^{-iH\Delta t} | \phi_1 \rangle \langle \phi_1 | \phi_a \rangle \end{aligned} \quad (21)$$

We make use of the following expressions

$$\langle \phi_1 | \phi_a \rangle = \delta(\phi_1 - \phi_a) \quad (22)$$

$$\langle \phi_{i+1} | \pi_i \rangle = \exp \left[i \int d^3x \pi_i(\vec{x}) \phi_{i+1}(\vec{x}) \right]. \quad (23)$$

For $\Delta t \rightarrow 0$ the exponential function can be expanded

$$\begin{aligned} \langle \pi_i | e^{-i\hat{H}\Delta t} | \phi_i \rangle &\simeq \langle \pi_i | (1 - \hat{H}\Delta t) | \phi_i \rangle \\ &= \langle \pi_i | \phi_i \rangle (1 - H_i \Delta t) \\ &= (1 - H_i \Delta t) \exp \left[i \int d^3x \pi_i(\vec{x}) \phi_i(\vec{x}) \right], \end{aligned} \quad (24)$$

where

$$H_i = \int d^3x \mathcal{H}(\pi_i(\vec{x}), \phi_i(\vec{x})). \quad (25)$$

Taken all expressions together yields

$$\begin{aligned} \langle \phi_a | e^{-i\hat{H}t_f} | \phi_a \rangle &= \lim_{N \rightarrow \infty} \int \left(\prod_{i=1}^N \frac{d\pi_i d\phi_i}{2\pi} \right) \delta(\phi_1 - \phi_a) \\ &\times \exp \left\{ -i\Delta t \sum_{j=1}^N \int d^3x [\mathcal{H}(\pi_j, \phi_j) - \right. \\ &\left. - \pi_j \frac{\phi_{j+1} - \phi_j}{\Delta t}] \right\} \end{aligned} \quad (26)$$

Here holds $\phi_{N+1} = \phi_a = \phi_1$. In the continuum limit we obtain

$$\begin{aligned} \langle \phi_a | e^{-i\hat{H}t_f} | \phi_a \rangle &= \int_{\phi(\vec{x},0)=\phi_a}^{\phi(\vec{x},t_f)=\pm\phi_a} \mathcal{D}\pi \int \mathcal{D}\phi \exp \left[i \int_0^{t_f} dt \int d^3\vec{x} \left(\pi \frac{\partial \phi}{\partial t} - \mathcal{H}(\phi, \pi) \right) \right] \\ &= \int_{\phi(\vec{x},0)=\phi_a}^{\phi(\vec{x},t_f)=\pm\phi_a} \mathcal{D}\pi \int \mathcal{D}\phi \exp \left[i \int_0^{t_f} dt \int d^3\vec{x} \mathcal{L}(\phi, \pi) \right] \end{aligned} \quad (27)$$

The notations $\mathcal{D}\pi$ and $\mathcal{D}\phi$ stand for the Functional Integration over fields and their conjugate momenta.

The partition function of a quantum statistical system is defined as

$$\begin{aligned} Z &= \text{Tr}\{e^{-\beta(\hat{H}-\mu_i\hat{N}_i)}\} \\ &= \int d\phi_a \langle \phi_a | e^{-\beta(\hat{H}-\mu_i\hat{N}_i)} | \phi_a \rangle \end{aligned} \quad (28)$$

The expression under the integral shows formal equivalence to the time evolution operator (27) except for the factor i in the exponent. The formal equivalence can be made still closer by introducing the imaginary time $\tau = it$. \hat{N}_i is an operator corresponding to conserved charges in the system, such as baryon number. These conservation laws can be incorporated into the formalism as constraints by the method of Lagrangian multipliers. This is done by the replacement

$$\mathcal{H}(\pi, \phi) \rightarrow \mathcal{K}(\pi, \phi) = \mathcal{H}(\pi, \phi) - \mu_\gamma \mathcal{N}_\gamma(\pi, \phi) \quad (29)$$

The result for the partition function reads

$$Z = \int [d\pi] \int_{\pm} [d\phi] \exp \left(\int_0^\beta d\tau \int d^3x \left(i\pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}(\pi, \phi) + \mu_\gamma \mathcal{N}_\gamma(\pi, \phi) \right) \right) \quad (30)$$

The index \pm stands for the symmetry of the fields at the borders of the imaginary time interval which is determined up to a phase factor $\phi(\vec{x}, 0) = \pm \phi(\vec{x}, \beta)$, where the upper sign stands for Bosons, while the lower is for Fermions.

The next step is the transition to the Fourier representation

$$\phi(\mathbf{x}, \tau) = \left(\frac{\beta}{V} \right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x} + \omega_n \tau)} \phi_n(\mathbf{p}) \quad (31)$$

which allows to get rid of differential operators in the action functional and transform it to an algebraic expression of momenta and frequencies. Due to the (anti)periodicity on the imaginary time interval the conjugate frequencies become discrete. For bosonic fields $\phi(\vec{x}, 0) = \phi(\vec{x}, \beta)$ is fulfilled for

$$\omega_n = 2n\pi T \quad (32)$$

The ω_n are denoted as *Matsubara-Frequencies*. For fermionic fields a similar argument leads to the fermionic Matsubara frequencies $\omega_n = (2n + 1)\pi T$.

2 Bosonic Fields

2.1 Neutral Scalar Field

The most general renormalizable Lagrangian for a neutral scalar field is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - U(\phi) , \quad (33)$$

where the potential is

$$U(\phi) = g\phi^3 + \lambda\phi^4 , \quad (34)$$

and $\lambda \geq 0$ for stability of the vacuum. The momentum conjugate to the field is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \frac{\partial \phi}{\partial t} , \quad (35)$$

and the Hamiltonian is

$$\mathcal{H} = \pi \frac{\partial \phi}{\partial t} - \mathcal{L} = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + U(\phi) . \quad (36)$$

There is no conserved charge.

The first step in evaluating the partition function is to return to the discretized version

$$\begin{aligned} Z = & \lim_{N \rightarrow \infty} \left(\prod_{i=1}^N \int_{-\infty}^{\infty} \frac{d\pi_i}{2\pi} \int_{\text{periodic}} d\phi_i \right) \\ & \exp \left\{ \sum_{j=1}^N \int d^3x \left[i\pi_j (\phi_{j+1} - \phi_j) - \Delta\tau \left(\frac{1}{2} \pi_j^2 + \frac{1}{2} (\nabla \phi_j)^2 + \frac{1}{2} m^2 \phi_j^2 + U(\phi_j) \right) \right] \right\} . \end{aligned} \quad (37)$$

The momentum integrations can be done immediately since they are just products of Gaussian integrals. Divide position space into M^3 little cubes with $V = L^3$, $L = aM$, $a \rightarrow 0$, $M \rightarrow \infty$, M an integer.

For convenience, and to ensure that Z remains explicitly dimensionless at each step in the calculation, we write $\pi_j = A_j / (a^3 \Delta\tau)^{1/2}$ and integrate A_j from $-\infty$ to $+\infty$. For each cube we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{dA_j}{2\pi} \exp \left[-\frac{1}{2} A_j^2 + i \left(\frac{a^3}{\Delta\tau} \right)^{1/2} (\phi_{j+1} - \phi_j) A_j \right] \\ & = (2\pi)^{-1/2} \exp \left[\frac{-a^3 (\phi_{j+1} - \phi_j)^2}{2\Delta\tau} \right] . \end{aligned} \quad (38)$$

Thus far we have

$$\begin{aligned} Z = & \lim_{M, N \rightarrow \infty} (2\pi)^{-M^3 N/2} \int \left[\prod_{i=1}^N d\phi_i \right] \\ & \exp \left\{ \Delta\tau \sum_{j=1}^N \int d^3x \left[-\frac{1}{2} \left(\frac{(\phi_{j+1} - \phi_j)}{\Delta\tau} \right)^2 - \frac{1}{2} (\nabla \phi_j)^2 - \frac{1}{2} m^2 \phi_j^2 - U(\phi_j) \right] \right\} \end{aligned} \quad (39)$$

Returning to the continuum limit, we obtain

$$Z = N' \int_{\text{periodic}} \mathcal{D}\phi \exp \left(\int_0^\beta d\tau \int d^3x \mathcal{L} \right). \quad (40)$$

The Lagrangian is expressed as a functional of ϕ and its first derivatives. The formula (38) expresses Z as a functional integral over ϕ of the exponential of the action in imaginary time. The normalization constant is irrelevant, since multiplication of Z by any constant does not change the thermodynamics.

Next, we turn to the case of noninteracting fields with $U(\phi) = 0$. Interactions will be discussed separately. Define

$$S = \int_0^\beta d\tau \int d^3x \mathcal{L} = -\frac{1}{2} \int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial\phi}{\partial\tau} + (\nabla\phi)^2 + m^2\phi^2 \right) \right]. \quad (41)$$

Integrating by parts and taking note of the periodicity of ϕ , we obtain

$$S = -\frac{1}{2} \int_0^\beta d\tau \int d^3x \phi \left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2 \right) \phi. \quad (42)$$

The field can be decomposed into a Fourier series according to

$$\phi(\vec{x}, \tau) = \left(\frac{\beta}{V} \right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} e^{i(\vec{p}\vec{x} + \omega_n\tau)} \phi_n(\vec{p}), \quad (43)$$

where $\omega_n = 2\pi nT$, due to the constraint of periodicity that $\phi(\vec{x}, \beta) = \phi(\vec{x}, 0)$ for all \vec{x} . The normalization of (42) is chosen conveniently so that each Fourier amplitude is dimensionless. Substituting (42) into (41), and noting that $\phi_{-n}(-\vec{p}) = \phi_n^*(\vec{p})$ as required by the reality of $\phi_n(\vec{p})$, we find

$$S = -\frac{1}{2} \beta^2 \sum_n \sum_{\vec{p}} (\omega_n^2 + \omega^2) \phi_n(\vec{p}) \phi_n^*(\vec{p}) \quad (44)$$

with $\omega = \sqrt{\vec{p}^2 + m^2}$. The integrand depends only on the amplitude of ϕ and not its phase. The phases can be integrated out to get

$$\begin{aligned} Z &= N' \Pi_n \Pi_{\vec{p}} \left[\int_{-\infty}^{\infty} dA_n(\vec{p}) \exp \left[-\frac{1}{2} \beta^2 (\omega_n^2 + \omega^2) A_n^2(\vec{p}) \right] \right] \\ &= N' \Pi_n \Pi_{\vec{p}} \left[2\pi / (\beta^2 (\omega_n^2 + \omega^2)) \right]^{1/2}. \end{aligned} \quad (45)$$

Ignoring an overall multiplicative factor independent of β and V , which does not affect the thermodynamics, we arrive at

$$Z = \Pi_n \Pi_{\vec{p}} \left[\beta^2 (\omega_n^2 + \omega^2) \right]^{-1/2}. \quad (46)$$

More formally one can arrive at this result by using the general rules for Gaussian functional integrals over commuting (bosonic) variables, derived before in the QFT chapter, since (39) and (41) can be expressed as

$$Z = N' \int \mathcal{D}\phi \exp \left[-\frac{1}{2}(\phi, D\phi) \right] = N' \text{constant} (\det D)^{-1/2}, \quad (47)$$

where $D = \beta^2(\omega_n^2 + \omega^2)$ in (\vec{p}, ω_n) space and $(\phi, D\phi)$ denotes the inner product on the function space.

Thus far we have

$$\ln Z = -\frac{1}{2} \sum_n \sum_{\vec{p}} \ln [\beta^2(\omega_n^2 + \omega^2)]. \quad (48)$$

Note that the sum over n is divergent. This unfortunate feature stems from our careless handling of the integration measure $\mathcal{D}\phi$. A more rigorous treatment using the proper definition of $\mathcal{D}\phi$ gives a finite result. In order to handle (47), we make use of

$$\ln [(2\pi n)^2 + \beta^2 \omega^2] = \int_1^{\beta^2 \omega^2} \frac{d\Theta^2}{\Theta^2 + (2\pi n)^2} + \ln [1 + (2\pi n)^2], \quad (49)$$

where the last term is β -independent and thus can be ignored. Furthermore,

$$\sum_{-\infty}^{\infty} \frac{1}{n^2 + (\Theta/2\pi)^2} = \frac{2\pi^2}{\Theta} \left(1 + \frac{2}{e^\Theta - 1} \right), \quad (50)$$

hence

$$\ln Z = - \sum_{\vec{p}} \int_1^{\beta\omega} d\Theta \left(\frac{1}{2} + \frac{1}{e^\Theta - 1} \right). \quad (51)$$

Carrying out the Θ integral, and throwing away a β -independent piece, we finally arrive at

$$\ln Z = V \int \frac{d^3 p}{(2\pi)^3} \left[-\frac{1}{2} \beta \omega - \ln(1 - e^{-\beta\omega}) \right], \quad (52)$$

from which we obtain immediately the well-known expression for the ideal Bose gas ($\mu = 0$), once we subtract the divergent expressions for the zero-point energy

$$E_0 = -\frac{\partial}{\partial \beta} \ln Z_0 = V \int \frac{d^3 p}{(2\pi)^3} \frac{\omega}{2}, \quad (53)$$

and for the zero-point pressure

$$P_0 = T \frac{\partial}{\partial V} \ln Z_0 = -\frac{E_0}{V}, \quad (54)$$

which are typical for the quantum field-theoretical treatment. With this subtraction the vacuum is defined as the state with zero energy and pressure.

3 Fermionic Fields

Dirac fermions are described by a four-spinor field ψ with a Lagrangian density

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\partial - m)\psi \\ &= \psi^\dagger \gamma^0 \left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma} \cdot \vec{\nabla} - m \right) \psi .\end{aligned}\quad (55)$$

The momentum conjugate to this field is

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial t)} = i\psi^\dagger , \quad (56)$$

because $\gamma^0\gamma^0 = 1$. Thus, somewhat paradoxically, ψ and ψ^\dagger must be treated as independent entities in the Hamiltonian formulation. The Hamiltonian density is found by standard procedures,

$$\mathcal{H} = \Pi \frac{\partial\psi}{\partial t} - \mathcal{L} = \psi^\dagger \left(i \frac{\partial}{\partial t} \right) \psi - \mathcal{L} = \bar{\psi}(-i\vec{\gamma} \cdot \vec{\nabla} + m)\psi , \quad (57)$$

and the partition function is

$$Z = \text{Tr} e^{-\beta(\hat{H} - \mu\hat{Q})} , \quad (58)$$

with the conserved charge $Q = \int d^3x \psi^\dagger \psi$. The functional integral representation reads

$$Z = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp \left[\int_0^\beta d\tau \int d^3x \psi^\dagger \left(-\gamma^0 \frac{\partial}{\partial \tau} + i\vec{\gamma} \cdot \vec{\nabla} - m + \mu\gamma^0 \right) \psi \right] \quad (59)$$

As with bosons, it is most convenient to work in (\vec{p}, ω_n) space instead of (\vec{x}, τ) space, i.e.,

$$\psi_\alpha(\vec{x}, \tau) = \left(\frac{\beta}{V} \right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} e^{i(\vec{p}\vec{x} + \omega_n \tau)} \tilde{\psi}_{\alpha;n}(\vec{p}) , \quad (60)$$

where now $\omega_n = (2n + 1)\pi T$ due to the antiperiodicity of the (Grassmannian) Fermion field at the borders of the fundamental strip $0 \leq \tau \leq \beta$ in the imaginary time, $\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta)$.

Now we are ready to evaluate the fermionic partition function (58),

$$\begin{aligned}Z &= \left[\Pi_n \Pi_{\vec{p}} \Pi_\alpha \int id\tilde{\psi}_{\alpha;n}^\dagger(\vec{p}) d\tilde{\psi}_{\alpha;n}(\vec{p}) \right] e^S , \\ S &= \sum_n \sum_{\vec{p}} i\tilde{\psi}_{\alpha;n}^\dagger(\vec{p}) D_{\alpha\rho} \tilde{\psi}_{\rho;n}(\vec{p}) , \\ D &= -i\beta \left[(-i\omega_n + \mu) - \gamma^0 \vec{\gamma} \cdot \vec{p} - m\gamma^0 \right] ,\end{aligned}\quad (61)$$

using our knowledge about Grassmannian integration of Gaussian functional integrals, resulting in

$$Z = \det D . \quad (62)$$

Employing the identity

$$\ln \det D = \text{Tr} \ln D , \quad (63)$$

and evaluating the determinant in Dirac space explicitly (Exercise !), one finds

$$\ln Z = 2 \sum_n \sum_{\vec{p}} \ln \left\{ \beta^2 \left[(\omega_n + i\mu)^2 + \omega^2 \right] \right\} . \quad (64)$$

Since both positive and negative frequencies have to be summed over, the latter expression can be put in a form analogous to the above expression in the bosonic case,

$$\ln Z = \sum_n \sum_{\vec{p}} \left\{ \ln \left[\beta^2 \left(\omega_n^2 + (\omega - \mu)^2 \right) \right] + \ln \left[\beta^2 \left(\omega_n^2 + (\omega + \mu)^2 \right) \right] \right\} . \quad (65)$$

In the further evaluation we can go similar steps as in the bosonic case, with two exceptions: (1) the presence of a chemical potential, splitting the contributions of particles and antiparticles; (2) the Matsubara frequencies are now odd multiples of πT , so that the infinite sum to be exploited reads

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2 \pi^2 + \Theta^2} = \frac{1}{\Theta} \left(\frac{1}{2} - \frac{1}{e^{\Theta} + 1} \right) . \quad (66)$$

Integrating over the auxiliary variable Θ , and dropping terms independent of β and μ , we finally obtain

$$\ln Z = 2V \int \frac{d^3 p}{(2\pi)^3} \left[\beta \omega + \ln(1 + e^{-\beta(\omega - \mu)}) + \ln(1 + e^{-\beta(\omega + \mu)}) \right] . \quad (67)$$

Notice that the factor 2 corresponding to the spin- $\frac{1}{2}$ nature of the fermions comes out automatically. Separate contributions from particles (μ) and antiparticles ($-\mu$) are evident. Finally, the zero-point energy of the vacuum also appears in this formula.

4 Gauge Fields

4.1 Quantizing the Electromagnetic Field

In this chapter we would like to understand how the Faddeev-Popov ghosts as introduced to eliminate divergences of the gauge freedom will act in the pure gauge theory to eliminate unphysical degrees of freedom and allow to derive the blackbody radiation law with the correct number of physical degrees of freedom.

We start with recalling the path integral formulation of QED for the photon field $A_\mu(x)$ with the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (68)$$

and the free action functional

$$S = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} . \quad (69)$$

This action is invariant under gauge transformations

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \omega(x) , \quad (70)$$

where $\omega(x)$ is a scalar function which parametrizes the gauge transformations. The momenta conjugate to the space components of $A^i(x)$ are, up to a sign, the components $E_i(x) = \mathbf{E}_i(\mathbf{x})$ of the electric field

$$\pi_i = -E_i = -F_{0i} , \quad (71)$$

while the magnetic field $\mathbf{B}(\mathbf{x})$ is

$$B_i = \varepsilon_{ijk} \partial_j A^k . \quad (72)$$

We work in an axial gauge, $A^3 = 0$ to be specific. The momenta π_1 and π_2 are independent variables; E_3 is not an independent variable, but it is a function of π_1 and π_2 , which may be computed from Gauss's law $\nabla \cdot \mathbf{E} = \mathbf{0}$. There are thus two dynamical variables A^1 and A^2 with conjugate momenta π_1 and π_2 . We define $\pi_3 = -E_3(\pi_1, \pi_2)$, but π_3 is not to be interpreted as a conjugate momentum. The partition function is written as a Hamiltonian path integral

$$Z = \int \mathcal{D}(\pi_1, \pi_2) \int_{A^i(0)=A^i(\beta)} \mathcal{D}(A^1, A^2) \exp \left[\int_0^\beta d^4x (i\pi_1 \partial_\tau A^1 + i\pi_2 \partial_\tau A^2 - \mathcal{H}) \right] , \quad (73)$$

where we have used the notation

$$\int_0^\beta d^4x = \int_0^\beta d\tau \int d^3x , \quad (74)$$

while the Hamiltonian density \mathcal{H} is

$$\mathcal{H} = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) = \frac{1}{2} (\pi_1^2 + \pi_2^2 + \mathbf{E}_3^2(\pi_1, \pi_2) + \mathbf{B}^2) \quad (75)$$

Equation (72) is then transformed by using

$$1 = \int \mathcal{D}\pi_3 \delta(\pi_3 + E_3(\pi_1, \pi_2)) , \quad (76)$$

and

$$\begin{aligned} \delta(\pi_3 + E_3(\pi_1, \pi_2)) &= \delta(\nabla \cdot \pi) \det \left(\frac{\partial(\nabla \cdot \pi)}{\partial \pi_3} \right) \\ &= \det \left[\partial_3 \delta^3(\mathbf{x} - \mathbf{y}) \right] \delta(\nabla \cdot \pi) . \end{aligned} \quad (77)$$

In the following step, one inserts an integral representation of $\delta(\nabla \cdot \pi)$

$$\delta(\nabla \cdot \pi) = \int \mathcal{D}\mathbf{A}_4 \exp \left[\mathbf{i} \int_0^\beta d^4 \mathbf{x} \mathbf{A}_4 (\nabla \cdot \pi) \right] , \quad (78)$$

where $A_4 = iA_0$, and we work in Euclidean space now: $x_\mu = (\mathbf{x}, \mathbf{x}_4) = (\mathbf{x}, \tau)$, $A_4 = (\mathbf{A}, \mathbf{A}_4)$. Performing the π - integration, we are left with

$$Z = \int \mathcal{D}(A_1, A_2, A_4) \det \left[\partial_3 \delta^3(\mathbf{x} - \mathbf{y}) \right] \exp \left[\int_0^\beta d^4 x \left(\frac{1}{2} (i\partial_\tau \mathbf{A} - \mathbf{i}\nabla \mathbf{A}_4)^2 - \frac{1}{2} \mathbf{B}^2 \right) \right] , \quad (79)$$

where $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, \mathbf{0})$. Note that the argument of the exponential is

$$\frac{1}{2} \mathbf{E}^2 - \frac{1}{2} \mathbf{B}^2 = \mathcal{L} . \quad (80)$$

The A - integration is rendered more aesthetic by inserting

$$1 = \int \mathcal{D}A_3 \delta(A_3) , \quad (81)$$

and the partition function assumes the form

$$Z = \int \mathcal{D}A^\mu \delta(A_3) \det \left[\partial_3 \delta^3(\mathbf{x} - \mathbf{y}) \right] \exp \left(\int_0^\beta d^4 x \mathcal{L} \right) . \quad (82)$$

The axial gauge $A_3 = 0$ is not a particularly convenient gauge to use for practical computations. Furthermore, it is not immediately apparent that (81) is a gauge invariant expression for Z .

Take an arbitrary gauge specified by $F = 0$, where F is some function of A^μ and its derivatives. For the gauge above, $F = A_3$. For this gauge, (81) is given by

$$Z = \int \mathcal{D}A^\mu \delta(F) \det \left(\frac{\partial F}{\partial \alpha} \right) \exp \left(\int_0^\beta d^4 x \mathcal{L} \right) . \quad (83)$$

Equation (82) is manifestly gauge invariant: \mathcal{L} is invariant, the gauge fixing factor times the Jacobian of the transformation $\delta(F) \det(\partial F/\partial \alpha)$ is invariant, and the integration is over all four components of the vector potential. Equation (82) reduces to (81) in the case of the axial gauge $A_3 = 0$. We know this is correct since it was derived from first principles in the Hamiltonian formulation of the gauge theory, $Z = \text{Tr} e^{-\beta \hat{H}}$

4.2 Blackbody radiation

It is important to verify that (82) describes blackbody radiation with two polarization degrees of freedom. We will do this here in the axial gauge $A_3 = 0$, the Feynman gauge is left as an exercise.

In the axial gauge, we rewrite (78) as

$$\begin{aligned} Z &= \int \mathcal{D}(A_0, A_1, A_2) \det(\partial_3) e^{S_0} \\ S_0 &= \frac{1}{2} \int d\tau \int d^3x (A_0, A_1, A_2) \\ &\quad \times \begin{pmatrix} \nabla^2 & -\partial_1 \frac{\partial}{\partial \tau} & -\partial_2 \frac{\partial}{\partial \tau} \\ -\partial_1 \frac{\partial}{\partial \tau} & \partial_2^2 + \partial_3^2 + \frac{\partial^2}{\partial \tau^2} & \partial_1 \partial_2 \\ -\partial_2 \frac{\partial}{\partial \tau} & -\partial_1 \partial_2 & \partial_1^2 + \partial_3^2 + \frac{\partial^2}{\partial \tau^2} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix}. \end{aligned} \quad (84)$$

We can express the determinant of ∂_3 as a functional integral over a complex ghost field C : that is, a Grassmann field with spin-0,

$$\det(\partial_3) = \int \mathcal{D}\bar{C} \mathcal{D}C \exp \left(\int_0^\beta d\tau \int d^3x \bar{C} \partial_3 C \right). \quad (85)$$

These ghost fields \bar{C} and C are not physical fields since they do not appear in the Hamiltonian. Furthermore, since they are anticommuting scalar fields they violate the connection between spin and statistics. It is simply a convenient functional integral representation of the determinant of an operator. The greatest applicability of these fictitious ghost fields will be to non-Abelian gauge theories, see also Sect. 5.2.1. of Le Bellac [2].

In frequency-momentum space the partition function is expressed as

$$\begin{aligned} \ln Z &= \ln \det(\beta p_3) - \frac{1}{2} \ln \det(D), \\ D &= \beta^2 \begin{pmatrix} \mathbf{p}^2 & -\omega_n p_1 & -\omega_n p_2 \\ -\omega_n p_1 & \omega_n^2 + p_2^2 + p_3^2 & -p_1 p_2 \\ -\omega_n p_1 & -p_1 p_2 & \omega_n^2 + p_1^2 + p_3^2 \end{pmatrix}. \end{aligned}$$

Carrying out the determinantal operation,

$$\begin{aligned} \ln Z &= \frac{1}{2} \text{Tr} \ln(\beta^2 p_3^2) - \frac{1}{2} \text{Tr} \ln \left[\beta^6 p_3^2 (\omega_n^2 + \mathbf{p}^2)^2 \right] \\ &= \ln \left(\Pi_n \Pi_{\mathbf{p}} \left[\beta^2 (\omega_n^2 + \mathbf{p}^2) \right]^{-1} \right) \\ &= 2V \int \frac{d^3p}{(2\pi)^3} \left[-\frac{1}{2} \beta \omega - \ln(1 - e^{-\beta \omega}) \right]. \end{aligned} \quad (86)$$

Here, $\omega = |\mathbf{p}|$. Comparison with the result for the scalar field case shows that (85) describes massless bosons with two spin degrees of freedom in thermal equilibrium; in other words, blackbody radiation.

5 Interacting Fermion Systems: Hubbard-Stratonovich Trick

So far we have dealt with free quantum fields in the absence of interactions and have obtained nice closed expressions for the thermodynamic potential, i.e., therefore also for the generating functionals of the thermodynamic Green functions. However, once we switch on the interactions in our model field theories, there is only a very limited class of soluble models, in general we have to apply approximations. The most common technique is based on perturbation theory, which requires a small parameter. For strong interactions at low momentum transfer (the infrared region), the coupling is nonperturbatively strong and alternative, nonperturbative methods have to be invoked. One of the strategies, which is especially suitable for the treatment of quantum field theories within the path integral formulation is based on the introduction of collective variables (auxiliary fields) by an exact integral transformation due to Stratonovich and Hubbard which allows to eliminate (integrate out) the elementary degrees of freedom. Generally, the (dual) coupling of the auxiliary fields is weak so that perturbative expansions of the nonlinear effective action make sense and provide useful results already at low orders of this expansion.

A general class of interactions for which the Hubbard-Stratonovich (HS) transformation is immediately applicable, are four-fermion couplings of the current-current type

$$\mathcal{L}_{int} = G(\bar{\psi}\psi)^2 . \quad (87)$$

A Fermi gas with this type of interaction serves as a model for electronic superconductivity (Bardeen-Cooper-Schrieffer (BCS) model, 1957) or for chiral symmetry breaking in quark matter (Nambu–Jona-Lasinio (NJL) model, 1961).

The HS-transformation for (86) reads

$$\exp \left[G(\bar{\psi}\psi)^2 \right] = \mathcal{N} \int \mathcal{D}\sigma \exp \left[\frac{\sigma^2}{4G} + \bar{\psi}\psi\sigma \right] \quad (88)$$

and allows to bring the functional integral over fermionic fields into a quadratic (Gaussian) form so that fermions can be integrated out. This is also called *Bosonization* procedure.

5.1 Nambu–Jona-Lasinio (NJL) Model

Here we will present an application of the HS technique to the NJL model for quark matter at finite densities and temperatures. This is possible since the interaction of this model is of the current-current form and therefore the HS trick for the bosonization of 4-fermion interactions applies. The Lagrangian density is given by

$$\mathcal{L} = \bar{q}_{i\alpha} (i\cancel{\partial}\delta_{ij}\delta_{\alpha\beta} - M_{ij}^0\delta_{\alpha\beta} + \mu_{ij,\alpha\beta}\gamma^0) q_{j\beta}$$