σ - ω model of nuclear matter (Path integral approach)

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Walecka model for dense nuclear matter (I)

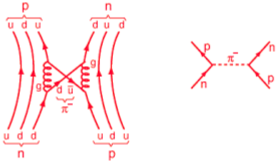
Meson exchange model

example: scalar (σ) meson

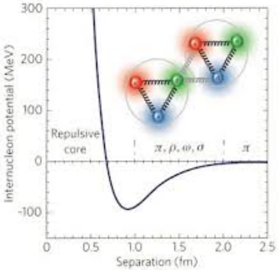
$$
\begin{aligned}
(-\triangle + m_\sigma^2)\sigma(\vec{r}) &= -g_\sigma \delta(\vec{r}) \\
\Rightarrow \quad \sigma(r) &= -\frac{g_\sigma}{4\pi}\frac{e^{-m_\sigma r}}{r} \\
V_{NN}^{(\sigma)}(r) &= g_\sigma \sigma(r) = -\frac{g_\sigma^2}{4\pi}\frac{e^{-m_\sigma r}}{r}
\end{aligned}
$$

Meson	I^π	T	S	M[MeV]
$\pi^0, \pi^\pm$	0^-	1	0	140
σ	0^+	0	0	$\approx$ 500
$K^0, K^\pm$	0^-	1/2	± 1	495
η	0^-	0	0	550
$\rho^0, \rho^\pm$	1^-	1	0	770
ω	1^-	0	0	780
δ	0^+	1	0	900

Feynman diagrams for π^- exchange



NN- Potential



Walecka model for dense nuclear matter (II)

Field theoretical formulation: Lagrangian and Path Integral for Partition Function

$$\mathcal{Z}_{gk}(T,V,\{\mu_i\}) = \int [d\overline{\Psi}][d\Psi] \exp\left\{ \int_0^{\beta=1/T} d\tau \int_V d^3\vec{x}\ \left(\mathcal{L}_0 + \mathcal{L}_I + \mu_p \Psi_p^+ \Psi_p + \mu_n \Psi_n^+ \Psi_n\right) \right\}$$

$$\mathcal{L}_0(\tau,\vec{x}) = \overline{\Psi}(\tau,\vec{x})\left(i\gamma_\mu \partial_\mu - m_N\right)\Psi(\tau,\vec{x})\ , \qquad \mathcal{L}_I(\tau,\vec{x}) = j_{\omega_\mu}(\tau,\vec{x})\frac{G_\omega}{2} j_{\omega_\mu}(\tau,\vec{x}) - j_\sigma(\tau,\vec{x})\frac{G_\sigma}{2} j_\sigma(\tau,\vec{x})$$

$$j_\sigma(\tau,\vec{x}) = \overline{\Psi}(\tau,\vec{x})\Psi(\tau,\vec{x})$$

$$j_{\omega_\mu}(\tau,\vec{x}) = \overline{\Psi}(\tau,\vec{x})\gamma_\mu\Psi(\tau,\vec{x})$$

$$\Psi = \begin{pmatrix} \psi_n \\ \psi_p \end{pmatrix}; \quad \psi_n = \begin{pmatrix} u_{n,\uparrow} \\ u_{n,\downarrow} \\ v_{n,\uparrow} \\ v_{n,\downarrow} \end{pmatrix} \begin{matrix} \}\ \text{Neutron} \\ \}\ \text{Antineutron} \end{matrix}$$

$\mu_n = \mu_p \quad \longrightarrow$ symmetric nuclear matter

$\mu_n \neq 0;\ \mu_p = 0 \quad \longrightarrow$ pure neutron matter

$\mu_n = \mu_p + \mu_{e^-} \quad \longrightarrow$ neutron star matter (β-equilibrium)

Walecka model for dense nuclear matter (III)

Evaluation of the Path Integral: Hubbard-Stratonovich trick

$$\exp\left(-\left(\overline{\Psi}\Psi\right)\frac{G_\sigma}{2}\left(\overline{\Psi}\Psi\right)\right) = \left(\det G_\sigma^{-1}\right)^{\frac{1}{2}}\int[d\sigma]\exp\left(\frac{\sigma^2}{2G_\sigma}+\sigma\overline{\Psi}\Psi\right)$$

Effective action quadratic $\Longrightarrow$ Gaussian Path Integral

$$\mathcal{S}\equiv\int_0^\beta d\tau\int d^3\vec{x}\,\overline{\Psi}(\vec{x},\tau)\left\{\left(-\gamma_0\frac{\partial}{\partial\tau}+i\vec{\gamma}\vec{\nabla}-m_N+\gamma_0\mu+\sigma-\gamma_\mu\omega_\mu\right)\Psi(\vec{x},\tau)+\frac{\sigma^2}{2G_\sigma}-\frac{\omega_\mu^2}{2G_{\omega_\mu}}\right\}$$

Fourier representation: $\Psi(\vec{x},\tau)=\sqrt{\frac{T}{V}}\sum_n\sum_{\vec{p}}e^{i(\vec{p}\vec{x}+\omega_n\tau)}\Psi_n(\vec{p})$, with $\omega_n\equiv\pi T(2n+1)$

$$\begin{aligned}
&\int_0^\beta d\tau\int d^3\vec{x}\,\overline{\Psi}(\vec{x},\tau)\left(-\gamma_0\frac{\partial}{\partial\tau}+i\vec{\gamma}\vec{\nabla}-m_N+\gamma_0\mu+\sigma-\gamma_0\omega_0\right)\Psi(\vec{x},\tau)\\
= \quad &\frac{1}{\beta V}\int_0^\beta d\tau\int d^3\vec{x}\sum_{n,n'}\sum_{\vec{p},\vec{p}'}\overline{\Psi}_{n'}(\vec{p}')\left(-i\gamma_0\omega_n-\vec{\gamma}\vec{p}-m_N^*+\gamma_0\mu^*\right)\Psi_n(\vec{p})\,\mathrm{e}^{i\left\{(\vec{p}-\vec{p}')\vec{x}+(\omega_n-\omega_{n'})\tau\right\}}\\
= \quad &\beta\sum_n\sum_{\vec{p}}\overline{\Psi}_n(\vec{p})(-\gamma_\mu p_\mu-m_N^*)\Psi_n(\vec{p})=\sum_n\sum_{\vec{p}}\overline{\Psi}_n(\vec{p})G^{-1}[\sigma,\omega_0]\Psi_n(\vec{p})
\end{aligned}$$

Effective mass $m_N^*=m_N-\sigma$, chemical potential $\mu^*=\mu-\omega_0$ and quasiparticle propagator

$$G^{-1}[\sigma,\omega] \quad = \quad -\beta(\gamma_\mu p_\mu+m_N^*)\quad ,\quad p_0=i\omega_n-\mu^*$$

Walecka model for dense nuclear matter (IV)

Evaluate fermionic Path Integral and mean field approximation:

$$\begin{aligned}
\mathcal{Z}_{gk}(T,V,\{\mu_i\}) &= \mathcal{N}\prod_{n,\vec{p}}\int[d\overline{\Psi}_n(\vec{p})][d\Psi_n(\vec{p})][d\sigma][d\omega_0]\mathrm{e}^{\left\{\frac{\sigma^2-\omega_0^2}{2G_{\omega_0}}+\sum_{n,\vec{p}}\overline{\Psi}_n(\vec{p})G^{-1}[\sigma,\omega_0]\Psi_n(\vec{p})\right\}} \\
&= \int[d\sigma][d\omega_0]\exp\left\{Tr\,\ln G^{-1}[\sigma,\omega_0]+\frac{\sigma^2}{2G_\sigma}-\frac{\omega_0^2}{2G_{\omega_0}}\right\} \\
&= \exp\left\{Tr\,\ln G^{-1}[\overline{\sigma},\overline{\omega}_0]+\frac{\overline{\sigma}^2}{2G_\sigma}-\frac{\overline{\omega}_0^2}{2G_{\omega_0}}\right\}
\end{aligned}$$

Stationarity condition: $\partial\ln Z_{gk}/\partial\overline{\sigma}=\partial\ln Z_{gk}/\partial\overline{\omega}_0=0$ corresponds to "gap equations":

$$\overline{\sigma} = -G_\sigma\,Tr\,G[\overline{\sigma},\overline{\omega}_0]=G_\sigma n_s\ ,\quad \overline{\omega}_0=-G_\omega\,Tr\,\gamma_0 G[\overline{\sigma},\overline{\omega}_0]=G_\omega n\ .$$

Thermodynamics: $\Omega(T,V,\mu)=-T\ln\mathcal{Z}_{gk}=-pV$

$$p(\mu,T)=\frac{1}{2}G_\omega n^2-\frac{1}{2}G_\sigma n_s^2+4T\int\frac{d^3\vec{p}}{(2\pi)^3}\left[\ln\left(1+e^{-\beta(E^*-\mu^*)}\right)+\ln\left(1+e^{-\beta(E^*+\mu^*)}\right)\right]$$

$$n=4\int\frac{d^3\vec{p}}{(2\pi)^3}\left[f_-(E^*)-f_+(E^*)\right]\ ,\ n_s=4\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{m_N^*}{E^*}\left[f_-(E^*)-f_+(E^*)\right]\ ,\ f_\pm(E^*)=\frac{1}{e^{\beta(E^*\mp\mu^*)}+1}$$

Quasiparticle properties $E^*=\sqrt{\vec{p}^2+{m_N^*}^2}$, $m_N^*=m_n-G_\sigma n_s$, $\mu^*=\mu-G_\omega n$.

Walecka model for dense nuclear matter (V)

Evaluate Traces: $Tr \ln G^{-1} = tr_p tr_D \ln G^{-1} = tr_p \ln \det_D G^{-1} = \sum_n \sum_{\vec{p}} \ln \det_D G^{-1}$

Scalar mean field

$$
\begin{aligned}
\overline{\sigma} &= -G_{\overline{\sigma}} Tr\, G[\overline{\sigma}, \overline{\omega}_0] \\
&= -2G_\sigma T \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} tr_D \left[\gamma_\mu p_\mu - (m - \overline{\sigma}) + i\gamma_0(\mu - \overline{\omega})\right]^{-1} \\
&= 2G_\sigma T \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{m^*}{\vec{p}^2 + m^{*2} + (\omega_n + i\mu^*)^2} \right) \\
&= G_\sigma \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m^*}{E^*} \left(\frac{1}{e^{\beta(E^* - \mu^*)} + 1} + \frac{1}{e^{\beta(E^* + \mu^*)} + 1} \right) \\
&\equiv G_\sigma n_s
\end{aligned}
$$

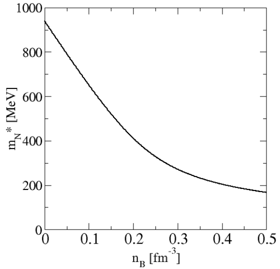
Vector mean field

$$
\begin{aligned}
\overline{\omega}_0 &= -G_{\overline{\omega}_0} Tr\, \gamma_0 G[\overline{\sigma}, \overline{\omega}_0] \\
&= G_\omega \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{1}{e^{\beta(E^* - \mu^*)} + 1} - \frac{1}{e^{\beta(E^* + \mu^*)} + 1} \right) \\
&\equiv G_\omega n
\end{aligned}
$$

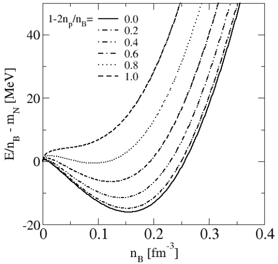
Matsubara sums $\longrightarrow$ Seminar!!

Walecka model for dense nuclear matter - results

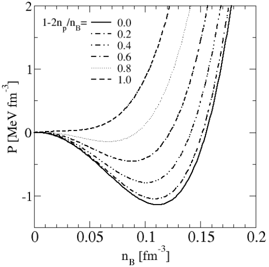
Effective mass



Energy per nucleon



Pressure



Symmetric nuclear matter ($n_p/n_B = 0.5$) saturates with a binding energy per nucleon of 16 MeV at $n_B = n_p + n_n = 0.16$ fm^{-3}. Increasing the asymmetry towards pure neutron matter ($n_p = 0$) makes the system unbound.

See, e.g., Kapusta's book "Finite temperature field theory" for the nuclear liquid-gas phase transition.