Modern Problems in Nuclear Physics III
(Introduction to Nuclear Matter under Extreme Conditions)

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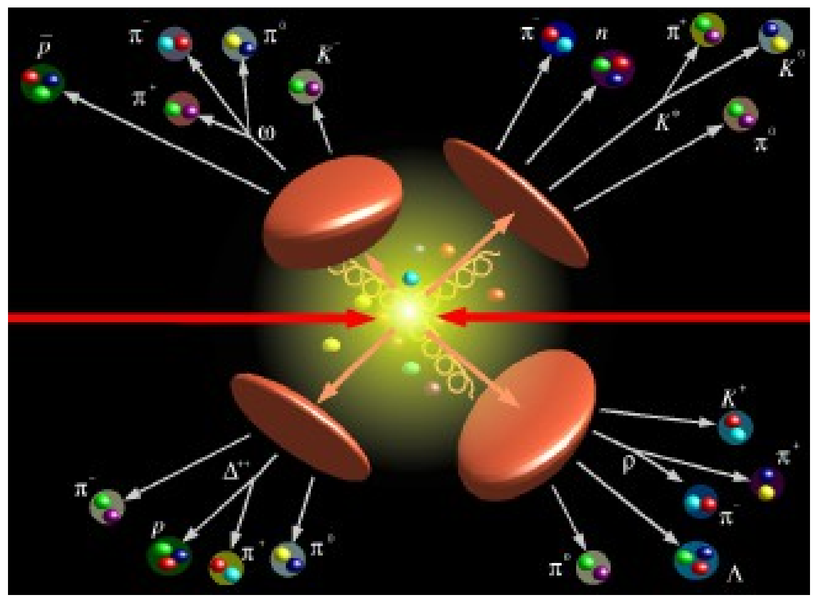
M. Srednicki,
"Chaos and Quantum Thermalization",
PRE 50 (1994) 888



F. Becattini @ ECT* 2014

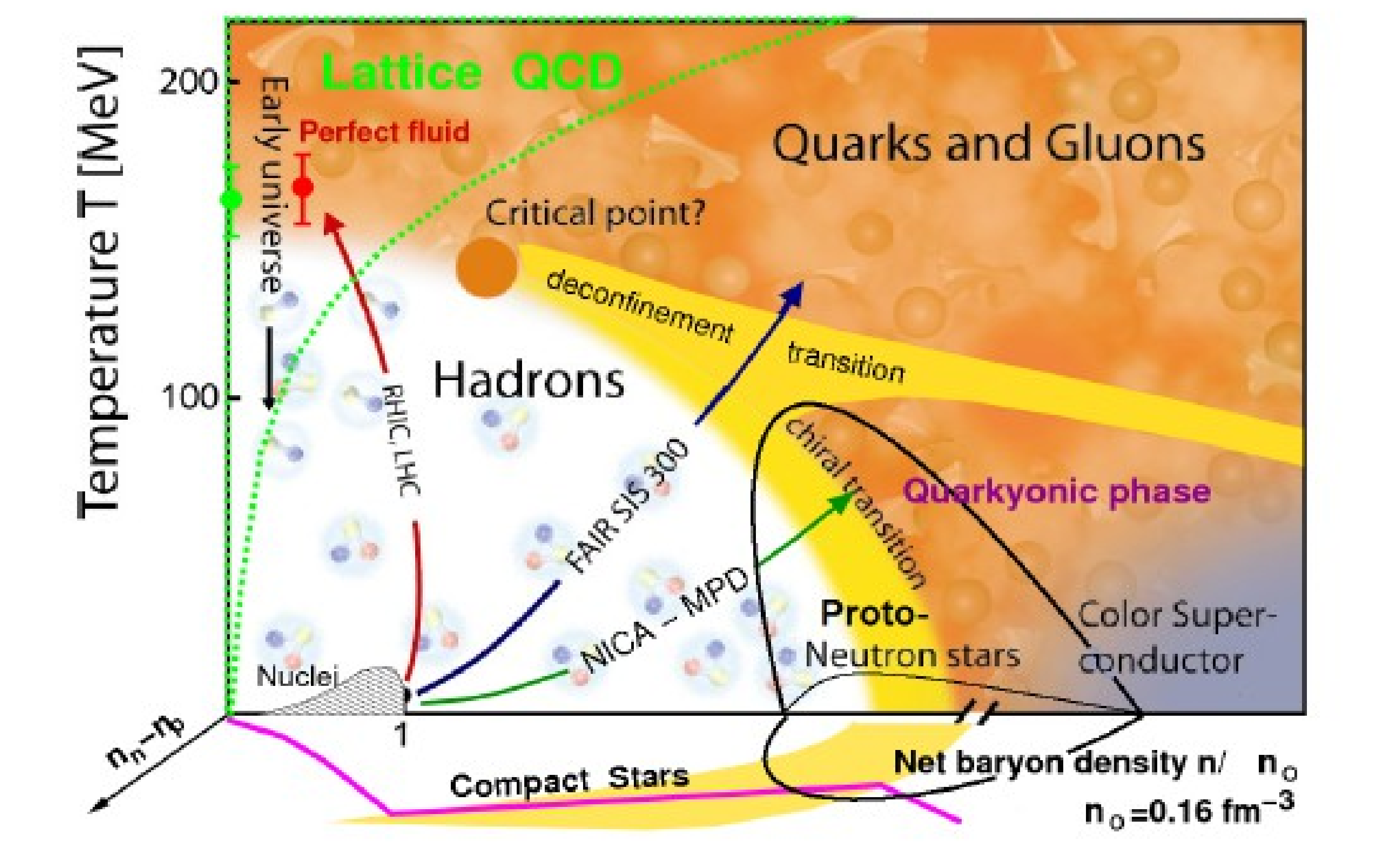
Part III: Nambu-Jona-Lasinio model: χSB & CSC, Sept. 21, 2016

DUBNA

Chiral phase transition, criticality and all that

Based on K. Yagi, T. Hatsuda, Y. Miake: "Quark-Gluon Plasma", CUP 2005

Possible (ρ, T) phase diagram of strongly interacting matter



Chiral phase transition, criticality and all that

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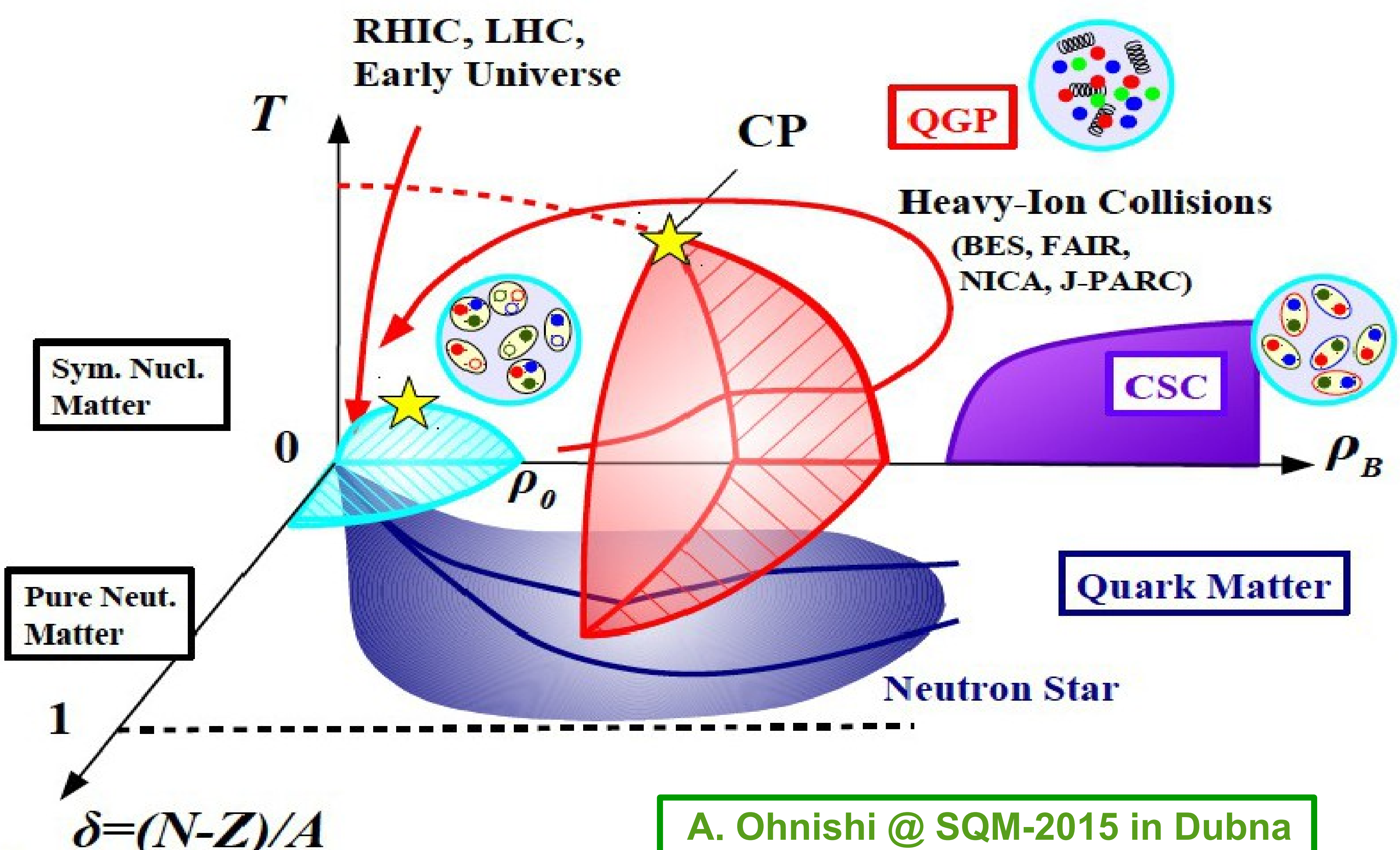


A. Ohnishi @ SQM-2015 in Dubna

Chiral phase transition

Order parameter?

Operator $\bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L$ not invariant under chiral rotation $SU_L(N_f) \times SU_R(N_f)$

$$\langle \bar{q}q \rangle = 0 : \text{the Wigner phase,}$$

$$\langle \bar{q}q \rangle \neq 0 : \text{the Nambu–Goldstone (NG) phase.}$$

Thermal expectation value $\langle \bar{q}q \rangle$ is a measure of dynamical breaking of chiral symmetry.
As temperature increases, quark pairing is dissociated by thermal fluctuations and the Transition from the NG phase to the Wigner phase takes place.
Analogy to electron pairing in metallic superconductors: **Nambu and Jona-Lasinio (1961)**

Questions:

1) What is the critical temperature of the chiral phase transition ?

2) What will be the order of the phase transition ?

3) What will be the observable phenomena associated with the chiral transition ?

Chiral phase transition

Order parameter $\langle \bar{q} q \rangle$ in hot/dense matter: Consider QCD partition function !

$$Z = \mathrm{Tr}\left[\mathrm{e}^{-\hat{K}_{\mathrm{QCD}}/T}\right] = \mathrm{e}^{-\Omega(T,V,\mu)/T} = \mathrm{e}^{P(T,\mu)V/T},$$

$$\hat{K}_{\mathrm{QCD}} = \hat{H}_{\mathrm{QCD}}(m_q = 0) + \sum_{q=\mathrm{u,d,s,\ldots}} \int d^3x \, \bar{q}(m_q - \mu_q \gamma_0) q,$$

$$\langle \bar{q} q \rangle = -\frac{\partial P(T,\mu)}{\partial m_q}.$$

High temperature expansion

$$P_q(T; m_q) = 4N_c \int \frac{d^3k}{(2\pi)^3} T \ln\left(1 + \mathrm{e}^{-E_q(k)/T}\right)$$

$$\simeq 4N_c \frac{7}{8}\left[\frac{\pi^2}{90}T^4 - \frac{1}{42}m_q^2 T^2 - \frac{1}{56\pi^2} m_q^4 \left(\ln\left(\frac{m_q^2}{(\pi T)^2}\right) + C\right) + \cdots\right]$$

where $E_q(k) = (k^2 + m_q^2)^{1/2}$ and $C = 2\gamma - 3/2 \simeq -0.346$, with $\gamma \simeq 0.577$ being the Euler constant.

Low temperature expansion $P(T) = P_\pi(T) + P_{\mathrm{vac}}$

$$\frac{\langle \bar{q} q \rangle}{\langle \bar{q} q \rangle_{\mathrm{vac}}} = 1 + \frac{1}{f_\pi^2} \frac{\partial P_\pi(T)}{\partial m_\pi^2}\bigg|_{m_\pi \to 0}$$

$$= 1 - \frac{T^2}{8 f_\pi^2} - \frac{1}{6}\left(\frac{T^2}{8 f_\pi^2}\right)^2 - \frac{16}{9}\left(\frac{T^2}{8 f_\pi^2}\right)^3 \ln\left(\frac{\Lambda_q}{T}\right) + O(T^8),$$

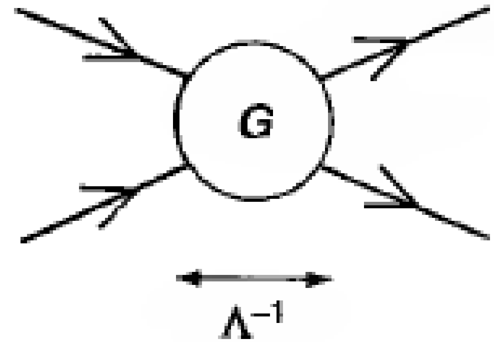
where $\Lambda_q (= 470 \pm 110\,\mathrm{MeV})$

$$f_\pi^2 m_{\pi^\pm}^2 = -\hat{m}\langle \bar{\mathrm{u}}\mathrm{u} + \bar{\mathrm{d}}\mathrm{d}\rangle_{\mathrm{vac}} + O(\hat{m}^2),$$

(Gell-Mann – Oakes – Renner)

The Nambu – Jona-Lasinio (NJL) Model

Simplest version, two-flavors:



$$\mathcal{L}_{\mathrm{NJL}} = \bar{q}(-i\gamma_\mu\partial_\mu + m)q - \frac{G^2}{2\Lambda^2}[(\bar{q}q)^2 + (\bar{q}i\gamma_5\boldsymbol{\tau}q)^2],$$

where ${}^t q(x) = (\mathrm{u}(x), \mathrm{d}(x))$ and $m = \mathrm{diag}(m_{\mathrm{u}}, m_{\mathrm{d}}) = m \cdot \mathbf{1}$

$$Z_{\mathrm{NJL}} = \int [d\bar{q}\ dq]\ \mathrm{e}^{-\int_0^{1/T} d\tau \int d^3x\ \mathcal{L}_{\mathrm{NJL}}}$$

$$= \int [d\bar{q}\ dq][d\Sigma]\ \mathrm{e}^{-\int_0^{1/T} d\tau \int d^3x \left[\bar{q}(-i\gamma\cdot\partial + m + G\Sigma)q + \frac{\Lambda^2}{2}\Sigma\Sigma^\dagger\right]}$$

$$\equiv \int [d\Sigma]\ \mathrm{e}^{-S_{\mathrm{eff}}(\Sigma;T)},$$

Hubbard–Stratnovich transformation

$$\mathrm{e}^{\frac{1}{2}y^2} = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}z^2 \pm zy}.$$

Grassmannian integration: det A = exp(Tr ln A)

$$S_{\mathrm{eff}}(\Sigma;T) = -\mathrm{Tr}\ \ln\,(-i\gamma\cdot\partial + m + G\Sigma) + \int_0^{1/T} d\tau \int d^3x \left(\frac{\Lambda^2}{2}\Sigma(x)\Sigma(x)^\dagger\right)$$

$$\Sigma(x)(= \sigma(x) + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x))$$

NJL Model, Meanfield approximation

Stationary solution: $\delta S_{\text{eff}} / \delta \Sigma(x) = 0$

Space-time independent and real scalar meanfield: $\Sigma(x) = \Sigma^{\dagger}(x) = \sigma.$

Gap equation: $\partial f_{\text{eff}} / \partial \sigma = 0,$ with $S_{\text{eff}}(\sigma; T) \equiv f_{\text{eff}}(\sigma; T) V/T.$

Dynamically generated fermion mass: $M = m + G\sigma$

Thermodynamic potential (free energy density):

Dispersion relation: $E(k) = \sqrt{k^2 + (m + G\sigma)^2}$

Quark degeneracy factor: $d_q \; (= 2_{\text{spin}} \times 2_{q\bar{q}} \times N_c \times N_f = 24)$

$$f_{\text{eff}}(\sigma; T) = \frac{\Lambda^2}{2}\sigma^2 + \int \frac{d^3k}{(2\pi)^3} \left[\frac{-d_q E(k)}{2} - d_q T \ln\left(1 + e^{-E(k)/T}\right) \right].$$

Interaction energy — Energy of quarks in Dirac sea — Entropy term

Interpretation: $f_{\text{eff}} = \varepsilon - Ts.$

Relation of minimum $\bar{\sigma}$ with dynamical quark mass and chiral condensate in the chiral limit (m=0): $M_q = G\bar{\sigma}, \quad \langle \bar{u}u + \bar{d}d \rangle = -\frac{\Lambda^2}{G}\bar{\sigma}.$

Dynamical symmetry breaking in NJL at T=0

Low energy model, restriction of momentum integral to |k|< Λ, and scale all dimensionful quantities by Λ, rewrite $f_{\rm eff}/\Lambda^4 \to f_{\rm eff}$

$$f_{\rm eff}(\sigma;0) \simeq -\frac{d_q}{16\pi^2} + \frac{1}{2}\left(\frac{1}{G^2} - \frac{1}{G_c^2}\right)(G\sigma)^2 + \frac{d_q}{64\pi^2}(G\sigma)^4 \ln\left(\frac{4}{(G\sigma)^2}\right) + O(\sigma^6),$$

We expanded the exact expression of free energy around σ ~ 0 and defined $G_c = \pi\sqrt{\frac{8}{d_q}}$.

$$G \le G_c \to \bar{\sigma} = 0: \ \text{the Wigner phase,}$$

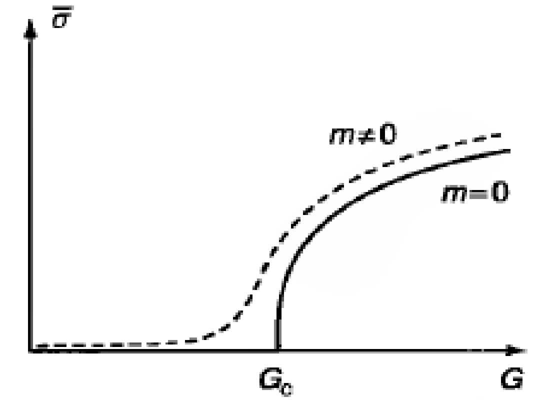
$$G > G_c \to \bar{\sigma} \neq 0: \ \text{the NG phase.}$$

Gap equation from d $f_{eff}(\sigma,0)$/d σ = 0 is given by:

$$\frac{G_c^2}{G^2} \simeq 1 - \frac{1}{2}(G\sigma)^2 \ln\left(\frac{4}{(G\sigma)^2 e}\right).$$

Solution is given by the Lambert function W(z), satisfying $W e^W = z$.
Asymptotic expansion near critical point gives (Exercise):

$$\bar{\sigma} \propto \sqrt{\frac{G^2 - G_c^2}{-\ln(G^2 - G_c^2)}} \qquad (G \searrow G_c).$$



Symmetry restoration at T ≠ 0

Using the high-temperature expansion one obtains:

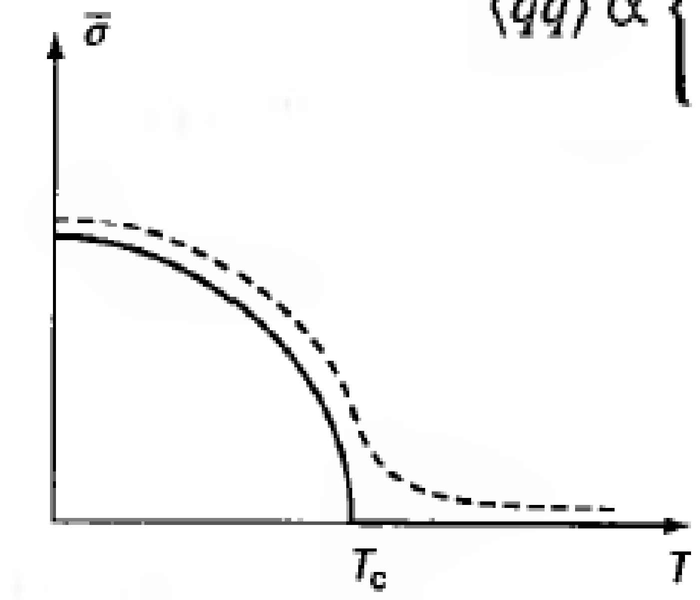
$$f_{\rm eff}(\sigma;T) = -\frac{d_q}{16\pi^2} - d_q\frac{7}{8}\frac{\pi^2}{90}T^4 + \frac{d_q}{48}(T^2 - T_c^2)(G\sigma)^2 + \frac{d_q}{64\pi^2}(G\sigma)^4\left[\ln\left(\frac{1}{(\pi T)^2}\right) + C\right] + O(\sigma^6),$$

With the critical temperature:

$$T_c = \sqrt{\frac{24}{d_q}\left(\frac{1}{G_c^2} - \frac{1}{G^2}\right)}.$$

The behaviour of the order parameter (chiral condensate) near $T \sim T_c$ can be found as:

$$\langle \bar{q}q \rangle \propto \begin{cases} 0 & (T \geq T_c), \\ -(T^2 - T_c^2)^{1/2} & (T < T_c). \end{cases}$$



For Λ >> σ , T we have (Exercise):

$$T_c \simeq \frac{\sqrt{3}}{\pi} M_0 = 0.55\ M_0.$$

For typical values of dynamical mass M_0~300-350 MeV, we obtain: $T_c \sim 165$ to $190\,\mathrm{MeV}$

Meanfield theory and Landau functional

Order of the phase transition

Consider the partition function in the thermodynamic limit, where $K=\{K_l\}$ is A set of generalized parameters, such as temperature, chem. Potential, coupling constants, external fields and so on.

$$Z = e^{-\Omega(K)} = e^{P(K)V},$$

Depending on the nature of the discontinuity of $\partial P(K)/\partial K_l$ across the phase boundary, The phase transition is either first order or continuous:

$$\text{first order,} \quad \text{if} \quad \frac{\partial P(K)}{\partial K_l} \quad \text{is discontinuous,}$$

$$\text{continuous,} \quad \text{if} \quad \frac{\partial P(K)}{\partial K_l} \quad \text{is continuous.}$$

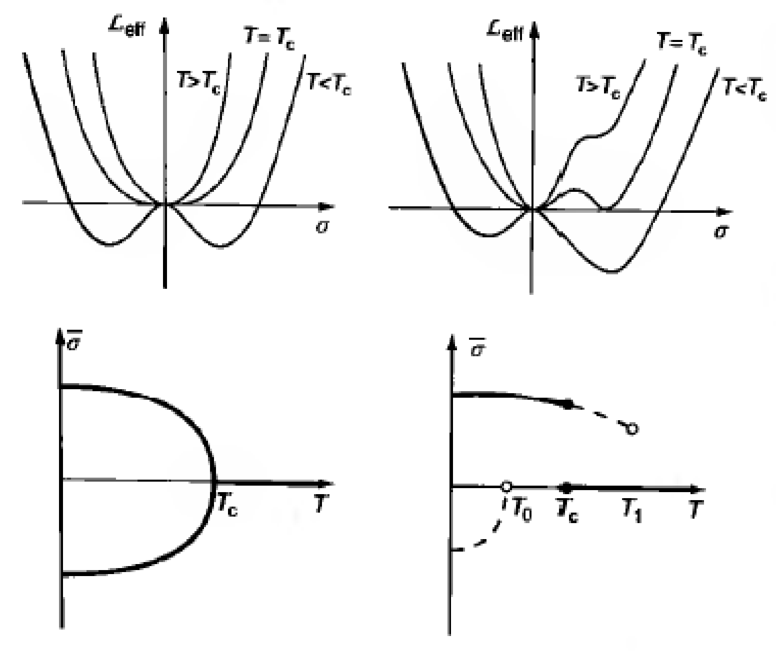
Let us consider an order parameter field σ(x), so that the partition function is:

$$Z = \int [d\sigma]\, e^{-S_{\text{eff}}(\sigma(x);K)},$$ where $S_{\text{eff}}(\sigma(x);K)$ is called the Landau functional.

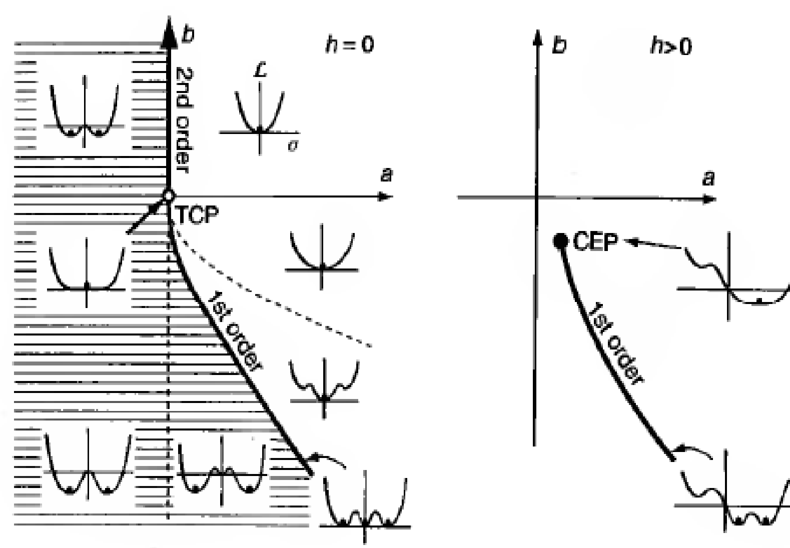
For a uniform system, we introduce the Landau function: $\mathcal{L}_{\text{eff}}(\sigma;K) = \sum_n a_n(K)\sigma^n,$

Landau functions

$\mathcal{L}_{\rm eff}$	Spin system in $d=3$	QCD
Fig. 6.3(a) Second order, $\frac{a}{2}\sigma^2+\frac{b}{4}\sigma^4-h\sigma$, controlled by $(a,\ h)$	Ising model $\begin{cases}\sigma\sim M\\ (a,h)\leftrightarrow(T,H)\end{cases}$	$N_c=3,\ N_f=2$ $\begin{cases}\sigma\sim\langle\bar{\rm u}{\rm u}+\bar{\rm d}{\rm d}\rangle\\ (a,h)\leftrightarrow(T,m_{\rm ud})\end{cases}$ $N_c=2,\ N_f=0$ $\begin{cases}\sigma\sim\langle L\rangle\\ (a,h)\leftrightarrow(T,1/m_{\rm Q})\end{cases}$
Fig. 6.3(b) First order, $\frac{a}{2}\sigma^2-\frac{c}{3}\sigma^3+\frac{b}{4}\sigma^4-h\sigma$, controlled by $(a,\ h)$	Z(3) Potts model $\begin{cases}\sigma\sim M\\ (a,h)\leftrightarrow(T,H)\end{cases}$	$N_c=3,\ N_f=3$ $\begin{cases}\sigma\sim\langle\bar{\rm u}{\rm u}+\bar{\rm d}{\rm d}+\bar{\rm s}{\rm s}\rangle\\ (a,h)\leftrightarrow(T,m_{\rm uds})\end{cases}$ $N_c=3,\ N_f=0$ $\begin{cases}\sigma\sim\langle L\rangle\\ (a,h)\leftrightarrow(T,1/m_{\rm Q})\end{cases}$
Fig. 6.5 Tricritical behavior, $\frac{a}{2}\sigma^2+\frac{b}{4}\sigma^4+\frac{c}{6}\sigma^6-h\sigma$, controlled by $(a,\ b,\ h)$	model for metamagnet $\begin{cases}\sigma\sim\tilde{M}\\ (a,b,h)\leftrightarrow(T,H,\tilde{H})\end{cases}$	$N_c=3,\ N_f=2+1$ $\begin{cases}\sigma\sim\langle\bar{\rm u}{\rm u}+\bar{\rm d}{\rm d}\rangle\\ (a,b,h)\leftrightarrow(T,m_{\rm s},m_{\rm ud})\\ (a,b,h)\leftrightarrow(T,\mu,m_{\rm ud})\end{cases}$



(a) (b)



(a) (b)

Meanfield theory and Landau function

Second order phase transition

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}a\sigma^2 + \frac{1}{4}b\sigma^4 - h\sigma,$$

$$a = a_t t \equiv a_t \frac{T - T_c}{T_c} \quad (a_t > 0),$$

$$b > 0, \;\; h \geq 0.$$

Stationarity condition: $a\sigma + b\sigma^3 = h$ Solutions: $\bar{\sigma}|_{h=0} = \begin{cases} 0 & (T \geq T_c) \\ \pm\left(-\frac{a}{b}\right)^{1/2} & (T < T_c). \end{cases}$

$$C_V(T, h=0) = -T \left.\frac{\partial^2 \mathcal{L}_{\text{eff}}(\bar{\sigma}; T)}{\partial T^2}\right|_{h=0} = \begin{cases} 0 & (T \geq T_c) \\ \frac{a_t^2}{2b}\frac{T}{T_c^2} & (T < T_c). \end{cases} \qquad \bar{\sigma}(T = T_c, h) = \left(\frac{h}{b}\right)^{1/3}$$

$$\chi_T(T, h)|_{h=0} = \left.\frac{\partial \bar{\sigma}}{\partial h}\right|_{h=0} = \begin{cases} \frac{1}{a} \sim |T - T_c|^{-1} & (T \geq T_c) \\ \frac{1}{-2a} \sim |T - T_c|^{-1} & (T < T_c), \end{cases}$$

Define critical exponents: $\bar{\sigma}(T \to T_c^-, h=0) \sim |T - T_c|^{\beta},$

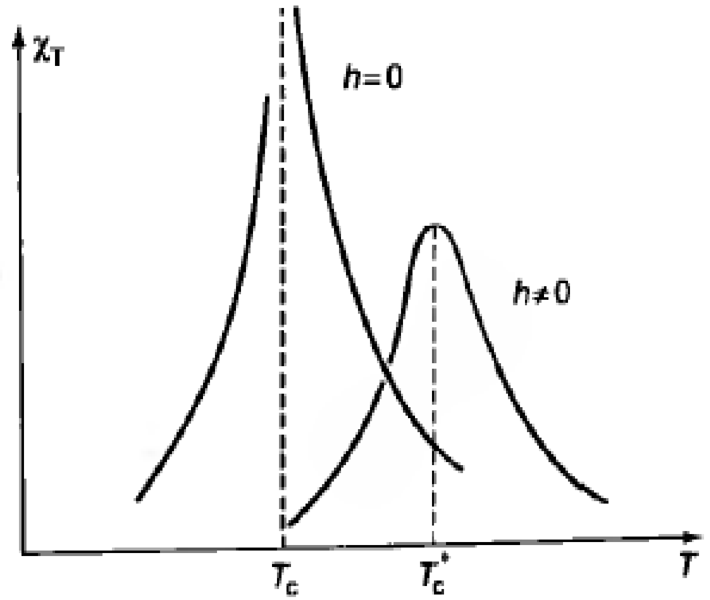
$$C_V(T \to T_c^{\pm}, h=0) \sim |T - T_c|^{-\alpha_{\pm}},$$

$$\bar{\sigma}(T = T_c, h \to 0) \sim h^{1/\delta},$$

$$\chi_T(T \to T_c^{\pm}, h=0) \sim |T - T_c|^{-\gamma_{\pm}},$$

Result:

$$\alpha_{\pm} = 0, \quad \beta = 1/2, \quad \gamma_{\pm} = 1, \quad \delta = 3.$$



The magnetic susceptibility for $h = 0$ and $h \neq 0$;

Meanfield theory, Landau functions

First oder transition, driven by cubic interaction:

Consider a Landau function, Eq. (6.33), with all the terms up to $n = 4$,

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}a\sigma^2 - \frac{1}{3}c\sigma^3 + \frac{1}{4}b\sigma^4 - h\sigma,$$

where we assume that

$$a = a_t t \equiv a_t \frac{T - T_0}{T_0} \quad (a_t > 0),$$

$$b > 0, \quad c > 0, \quad h \geq 0.$$

Stationary solutions of $\mathcal{L}_{\text{eff}}$ at $h = 0$ are simply obtained as

$$\bar{\sigma} = 0 \quad \text{and} \quad \bar{\sigma} = \frac{c \pm \sqrt{c^2 - 4ab}}{2b}.$$

discontinuous jump at $T = T_c$, or equivalently at $a = a_c \equiv (2c^2/9b)$

Meanfield theory, Landau functions

Tricritical behaviour with sextet interaction:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}a\sigma^2 + \frac{1}{4}b\sigma^4 + \frac{1}{6}c\sigma^6 - h\sigma,$$

$c > 0$ so that $\mathcal{L}_{\text{eff}}$ is bounded from below for large $|\sigma|$.

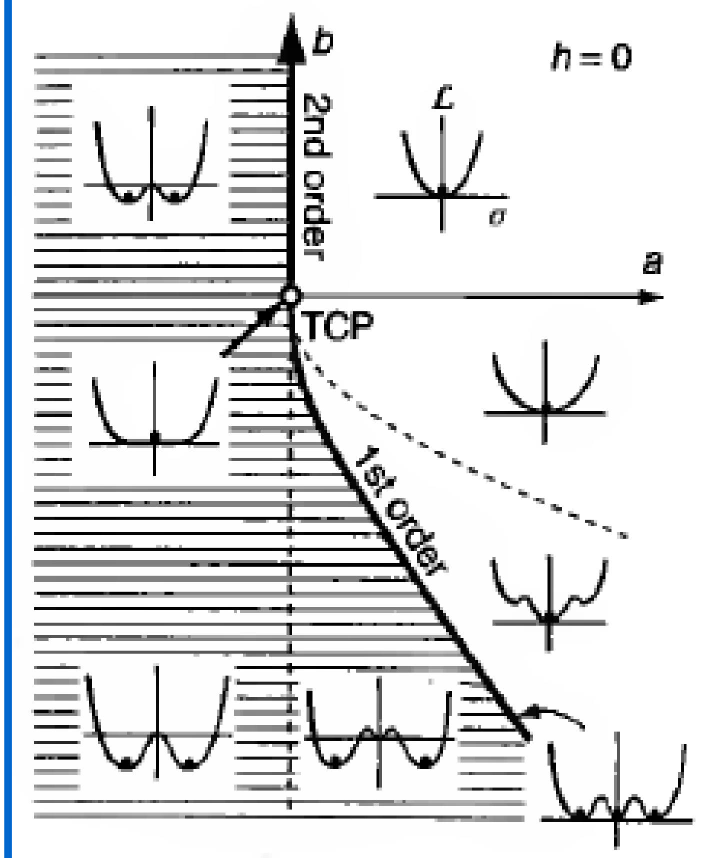
Both, *a* and *b* can change sign and may be parametrized as

$$a = a_t t + a_s s, \quad b = b_t t + b_s s,$$

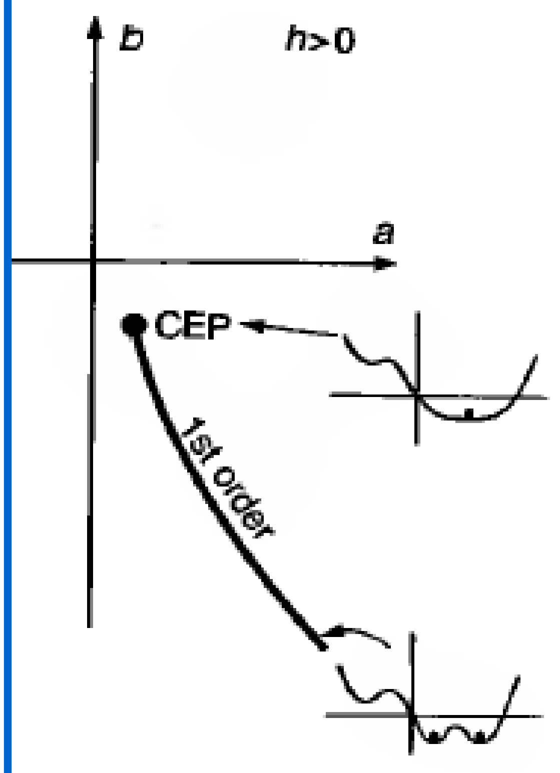
$$t = \frac{T - T_c}{T_c}, \quad s = \frac{S - S_c}{S_c}.$$

The point (*a,b*)=(0,0) is called the tricritical point (TCP), behaviour around it governed by *t*, reduced temperature, and an independent parameter *s*.
Three stationary solutions for *h=0* (Fig. a) are given by:

$$\bar{\sigma} = 0, \quad \bar{\sigma}^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$



(a)



(b)

For *h≠0* (Fig. b) the second order critical line disappears and the point where the first order line terminates is called critical end point (CEP).

Meanfield theory, Landau functions

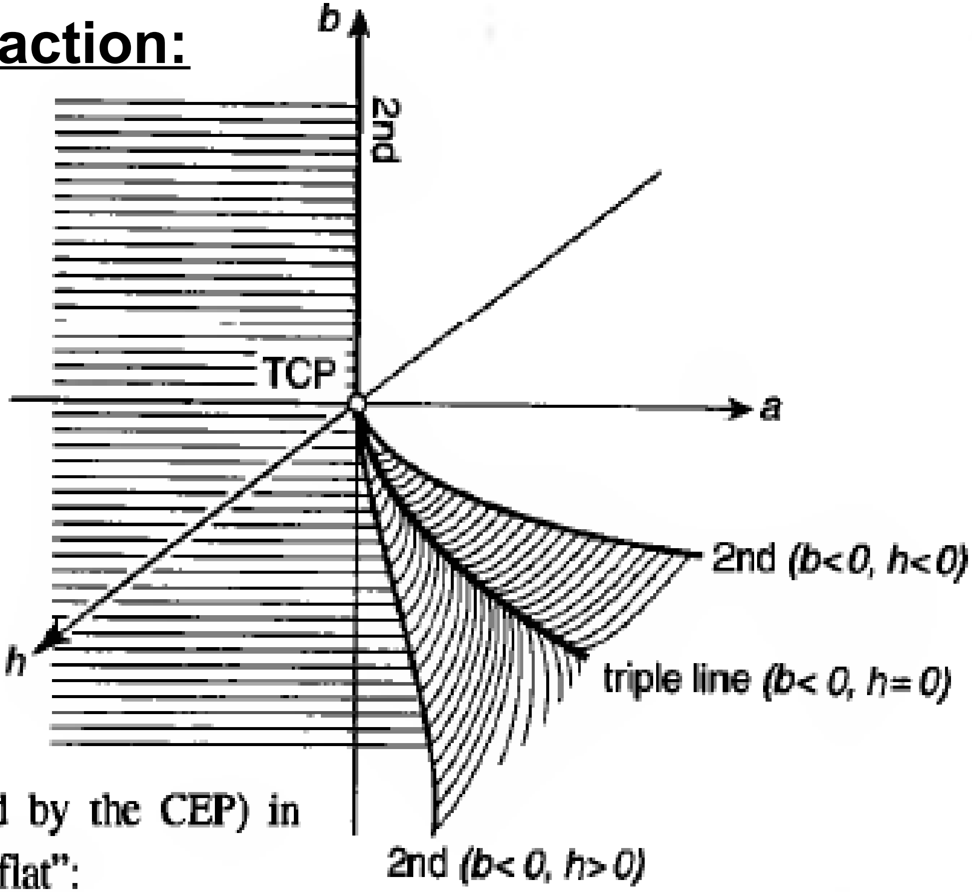
<u>Tricritical behaviour with sextet interaction:</u>

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}a\sigma^2 + \frac{1}{4}b\sigma^4 + \frac{1}{6}c\sigma^6 - h\sigma,$$

Three-dimensional parameter space (Figure), with a TCP at (a, b, h)=(0, 0, 0)

One second order line for (a, b, h)=(0, b>0, 0)

Two second order lines for:
(a, b, h)=(0, b<0, h>0)
(a, b, h)=(0, b<0, h<0)
Starting from the TCP



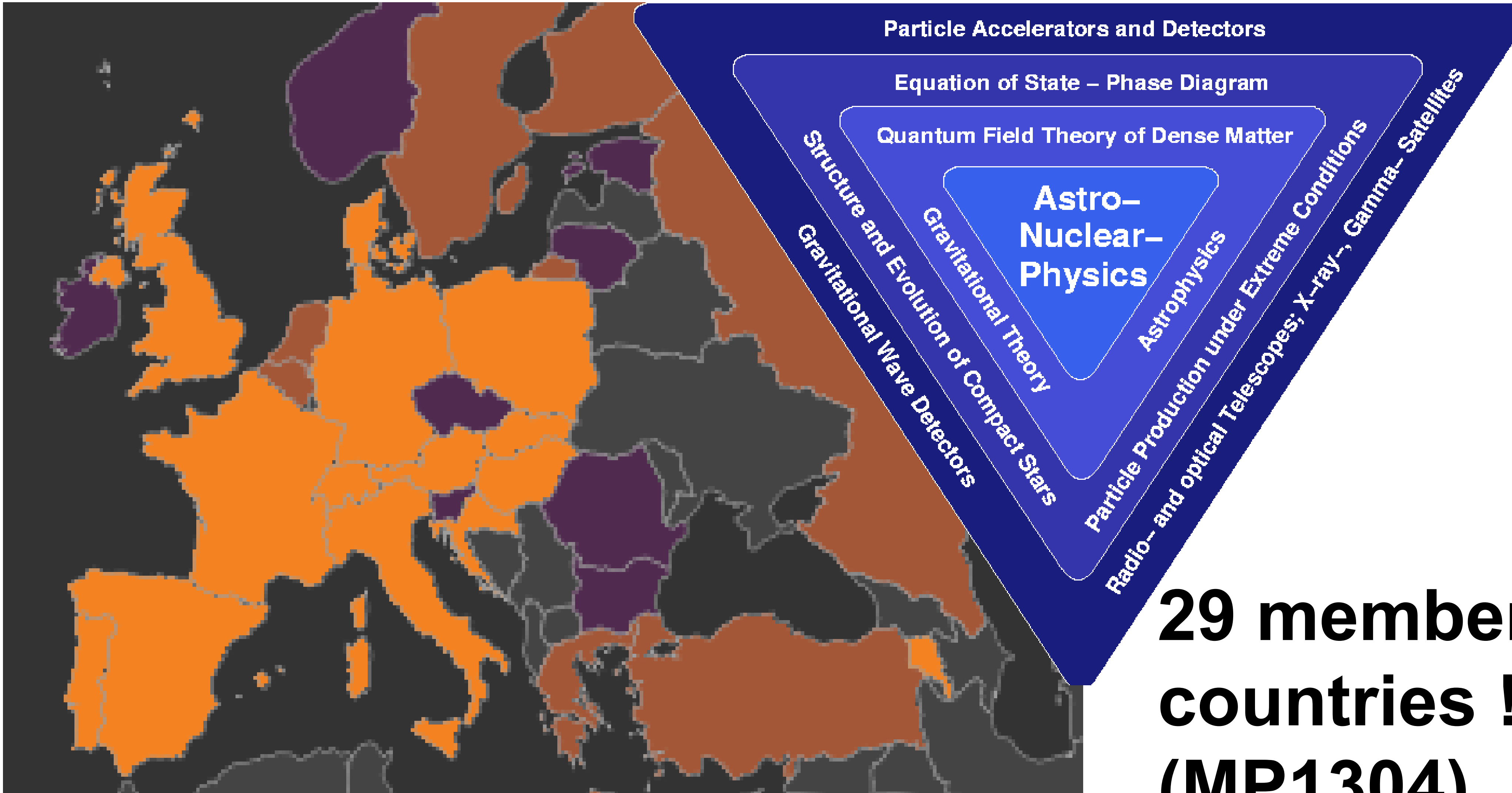
The edge of the wing (second order critical lines formed by the CEP) in (a, b, h)-space can be identified by the condition that $\mathcal{L}_{\text{eff}}$ is "flat":

$$\frac{\partial^n \mathcal{L}_{\text{eff}}}{\partial \sigma^n} = 0 \quad (n = 1, 2, 3), \quad \text{which may be rewritten as} \quad a = 5c\sigma^4, \quad b = -\frac{10}{3}c\sigma^2, \quad h = \frac{8}{3}c\sigma^5.$$

Eliminating σ, we have

$$\pm h = \frac{8c}{3}\left(\frac{a}{5c}\right)^{5/4} = \frac{8c}{3}\left(\frac{-3b}{10c}\right)^{5/2} \quad (a \geq 0, b \leq 0).$$

One may define the n-critical point
By introducing terms up to σ^n
In the Landau function



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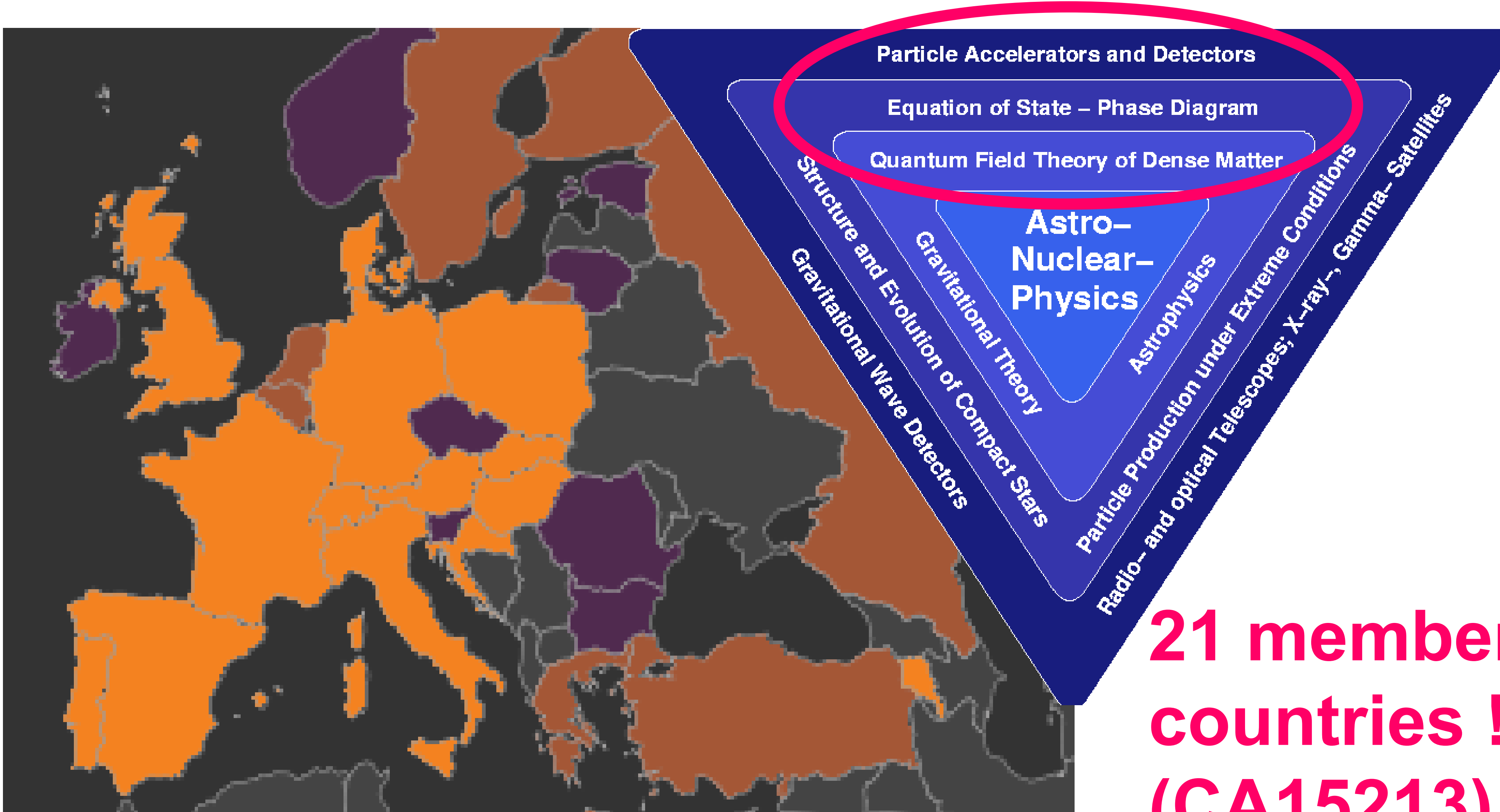

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