MEPhI Lecture Blaschke/Roepke: "Modern Problems in Nuclear Physics" - Exercise sheet 2 - 14.10.2020

1.1 Prove the key formula for path integrals over fermion fields which in its discretized form reads

$$\int d\eta_1^\dagger d\eta_1 \dots d\eta_N^\dagger d\eta_N \mathrm{e}^{\eta^\dagger D\eta} = \det D \ ,$$

where D is an $N \times N$ matrix and η_i, $i = 1$ to N is a set of Grassmann variables representing a fermion field configuration, $\eta_i^\dagger$ being the paired set. The algebra is defined by $\{\eta_i, \eta_j\} = \{\eta_i, \eta_j^\dagger\} = \{\eta_i^\dagger, \eta_j^\dagger\} = 0$.

Hints:
Consider a single Grassmann variable η which fulfills the anticommutator relation $\{\eta, \eta\} = 0$. Because of this the most general function of η is $f(\eta) = a + b\eta$, where a and b are $c-$ numbers, and integration is defined by $\int d\eta = 0$, $\int d\eta\ \eta = 1$.

The most general function of the pair of sets of Grassmann variables may be written as

$$f = a + \sum_i a_i\eta_i + \sum_i b_i\eta_i^\dagger + \sum_{i,j} a_{ij}\eta_i\eta_j + \sum_{i,j} b_{ij}\eta_i^\dagger\eta_j^\dagger + \sum_{i,j} c_{ij}\eta_i^\dagger\eta_j^\dagger + \dots + d\ \eta_1^\dagger\eta_1\eta_2^\dagger\eta_2\dots\eta_N^\dagger\eta_N$$

Integration over all variables with rules of Grassmann integration results in $\int d\eta_1^\dagger d\eta_1 \dots d\eta_N^\dagger d\eta_N\ f = d$.

1.2 The partition function of a Fermi-gas is given by $Z = \det D$, where $D = -i\beta\gamma^0[\gamma^0(-i\omega_n + \mu) - \vec{\gamma}\cdot\vec{p} - m]$. Use the representation of the 4x4 Dirac gamma matrices and evaluate the determinant to show the result:

$$\det_{\text{Dirac}} D = \beta^4[(\omega_n + i\mu)^2 + \omega^2(p)]^2 \ ,$$

where $\omega^2(p) = p^2 + m^2$.

1.3 The pressure of a noninteracting Fermi-gas is given by ($\beta = 1/T$)

$$p(T, V, \mu) = \frac{1}{\beta V}\ln Z_0(T, V, \mu) = \frac{2}{\beta V}\sum_n\sum_p \ln\{\beta^2[(\omega_n + i\mu)^2 + \omega^2(p)]\} \ .$$

perform the summation over the Matsubara- Frequencies $\omega_n = (2n + 1)\pi T$, and show the result

$$p(T, V, \mu) = 2\int \frac{d^3p}{(2\pi)^3}\{\omega(p) + T\ln[1 + \mathrm{e}^{-\beta(\omega(p)-\mu)}] + T\ln[1 + \mathrm{e}^{-\beta(\omega(p)+\mu)}]\}$$

1.4 Evaluate the pressure of a massless ideal Fermi-gas (e.g. Neutrinos) using the result of **1.3**. Remove the vacuum term and by performing the momentum integration show the result:

$$p = -\Omega = \frac{\mu^4}{12\pi^2} + \frac{\mu^2T^2}{6} + \frac{7\pi^2T^4}{180} \ .$$

Use the relation $\varepsilon = Ts - p + \mu n$ with $s = \partial\Omega/\partial T$ and $n = \partial\Omega/\partial\mu$ to prove $\varepsilon = 3p$ for this case.

2.1. The quantum statistical partition function. Evaluate the quantum statistical partition function for a free scalar field theory $\mathcal{L}_0(\phi) = -\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 - (\nabla\phi)^2 + m^2\phi^2\right]$ in the path integral representation

$$Z[T] = N\int_{\text{periodic}} \mathcal{D}\phi \exp\left(\int_0^\beta d\tau \int d^3x\mathcal{L}(\phi)\right) \ ,$$

where $\tau = it$ is an imaginary time and the fields have obey periodicity: $\phi(0, \vec{x}) = \phi(\beta, \vec{x})$.

1. Show that the result coincides with the partition function for a relativistic Bose gas

$$Z_0[T] = \exp\left\{ -\frac{1}{2} \sum_n \sum_{\vec{p}} \ln\left[\beta^2(\omega_n^2 + \omega^2(\vec{p})\right] \right\} ,$$

   where $\omega_n = 2n\pi T$ are the bosonic Matsubara frequencies and $\omega^2(\vec{p}) = \vec{p}^2 + m^2$ is the squared relativistic dispersion relation for the scalar particle of mass m.

2. Evaluate the sum over the Matsubara index n and obtain the thermodynamical potential for the massive, ideal Bose gas

$$\Omega_0(T) = -\frac{T}{V} \ln Z_0[T] = \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2}\omega(p) + T \ln(1 - \mathrm{e}^{-\beta\omega(p)}) \right] .$$

3. Remove ('renormalize') the divergent contribution due to the zero-point energy and evaluate the remaining thermal contribution to the pressure $P_0(T) = -\Omega_0(T)$ of the ideal Bose gas for the case $m/T \ll 1$ (high temperature limit) with the result

$$P_0(T) = \frac{\pi^2}{90} T^4 - \frac{1}{24} m^2 T^2 + \mathcal{O}(m^3) .$$

2.2. Quantizing the photon field. Blackbody radiation. In the axial gauge $A_3 = 0$, the partition function for the photon gas reads

$$\begin{aligned} Z &= \int \mathcal{D}(A_0, A_1, A_2) \det(\partial_3) \mathrm{e}^{S_0} \\ S_0 &= \frac{1}{2} \int d\tau \int d^3x (A_0, A_1, A_2) \\ &\times \begin{pmatrix} \nabla^2 & -\partial_1 \frac{\partial}{\partial\tau} & -\partial_2 \frac{\partial}{\partial\tau} \\ -\partial_1 \frac{\partial}{\partial\tau} & \partial_2^2 + \partial_3^2 + \frac{\partial^2}{\partial\tau^2} & \partial_1\partial_2 \\ -\partial_2 \frac{\partial}{\partial\tau} & -\partial_1\partial_2 & \partial_1^2 + \partial_3^2 + \frac{\partial^2}{\partial\tau^2} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} . \end{aligned} \qquad (1)$$

We can express the determinant of ∂_3 formally as a functional integral over a complex Grassmann field C with spin-0 (ghost field),

$$\det(\partial_3) = \int \mathcal{D}\bar{C}\mathcal{D}C \exp\left(\int_0^\beta d\tau \int d^3x \bar{C} \partial_3 C \right) . \qquad (2)$$

Carry out the path integrals and obtain the result

$$\ln Z = 2V \int \frac{d^3p}{(2\pi)^3} \left[-\frac{1}{2}\beta\omega - \ln(1 - \mathrm{e}^{-\beta\omega}) \right] , \qquad (3)$$

where $\omega = |\mathbf{p}|$. Discuss the role of the ghost fields in establishing the physical number of two spin degrees of freedom for the photon field.

3. Project: Grand canonical thermodynamic potential for quark matter. The grand canonical thermodynamic potential for quark matter within the nonlocal chiral quark model is given by

$$\begin{aligned} \Omega(\phi; \mu, T) &= \frac{\phi^2 - \phi_0^2}{4\, G_1} - \frac{6}{\pi^2} \int_0^\infty dq\, q^2 \Big\{ E_\phi(q) - E_{\phi_0}(q) \\ &+ T \ln\left[1 + \exp\left(-\frac{E_\phi(q) - \mu}{T} \right) \right] + T \ln\left[1 + \exp\left(-\frac{E_\phi(q) + \mu}{T} \right) \right] \Big\} . \end{aligned} \qquad (4)$$

The dispersion relation is $E_\phi(q) = \sqrt{q^2 + (m + \phi\, g(q))^2}$ where the mass gap (order parameter, chiral gap) $\phi = \phi(T, \mu)$ depends on temperature T and chemical potential μ and $\phi_0 = \phi(0, 0)$ is the value of the order parameter in the vacuum at $T = \mu = 0$. The coupling constant is $G_1 = 3.881/\Lambda^2$ where $\Lambda = 891$ MeV is the range of the separable interaction with the Gaussian formfactor $g(q) = \exp(-q^2/\Lambda^2)$ and $m = 2.177$ MeV is the current quark mass.

3.1 Find the $T = 0$ limit of Eq. (1) and derive the condition for the minimum of the thermodynamical potential with respect to a variation of the order parameter ϕ (gap equation)! Solve this gap equation for $\mu = 0$ in order to find the value of ϕ_0!

3.2 Solve the gap equation $\phi(\mu)$ for $0 \leq \mu \leq 500$ MeV and find the critical chemical potential μ_c where the mass gap jumps to a very low value (chiral symmetry restoration).

3.3 Insert the solution of the gap equation $\phi(\mu)$ in Eq. (1) and evaluate the pressure $p(\mu) = -\Omega(\phi; \mu, T = 0)$ by momentum integration for $0 \leq \mu \leq 500$ MeV.

3.4 Perform a fit of the quark matter pressure for $\mu > \mu_c$ to a bag model

$$p_{\mathrm{Bag}}(\mu) = \frac{\mu^4}{2\pi^2} - B \tag{5}$$

by adjusting the value of the bag constant B.

3.5 Perform the same steps for Lorentzian and cut-off (Nambu–Jona-Lasinio) form factor models! The parameters can be taken from the reference
`http://arXiv.org/abs/hep-ph/0602238`.
What is the influence of the form of the interaction on the structure of a neutron star with quark matter core?