Modern Problems of Nuclear Physics II - Exercise sheet 3 - 17.03.2021

3.1 Star of uniform energy density.

Integrate the Tolman-Oppenheimer-Volkoff (TOV) equations

$$\frac{dp(r)}{dr} = -\frac{[p(r)+\varepsilon(r)][M(r)+4\pi r^3 p(r)]}{r[r-2M(r)]} \,, \tag{1}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \varepsilon(r) \,, \tag{2}$$

for the case of a star with uniform energy density, $\varepsilon(r) = \varepsilon_0 = \text{constant}$. The radius R of the star is defined by $p(R) = 0$ and its mass is $M = M(R)$. Show that

a) the pressure profile is given by $p(r) = \varepsilon_0[\sqrt{1-2M/R} - \sqrt{1-2Mr^2/R^3}]/[\sqrt{1-2Mr^2/R^3} - 3\sqrt{1-2M/R}]$,

b) for the central pressure $p_c = p(0)$ holds the relation $2M/R = 1 - (p_c+\varepsilon_0)^2/(3p_c+\varepsilon_0)^2$,

c) the maximal value for the compactness of the star is $M/R = 4/9$,

d) the radius for this limiting compact star is $R_{\text{lim}} = 1/\sqrt{3\pi\varepsilon_0}$

3.2 Sequence of stars with constant energy density profile.

Integrating the second Tolman-Oppenheimer-Volkoff (TOV) equation with $\varepsilon(r) = \varepsilon_0\Theta(R-r)$ results in

$$M = 4\pi \int_0^R dr\, r^2 \varepsilon_0 = \frac{4\pi}{3} R^3\ \varepsilon_0 \,. \tag{3}$$

a) Assume that for a relevant range of neutron star masses ($M = 1...2\ M_\odot$) the radius R is constant. How does the mass depend on the (central) energy density ε_0 ?

b) Assume that the mass-radius dependence is bell-shaped:

$$M(R) = M_{\text{min}} + (M_{\text{max}} - M_{\text{min}})\frac{R_0^2}{(R-R_{\text{max}})^2 + R_0^2} \,. \tag{4}$$

Find the dependence $\varepsilon(M)$ in this case. Draw the curve $M(\varepsilon_0)$ for $M_{\text{max}} = 2\ M_\odot$, $M_{\text{min}} = 0.2\ M_\odot$, $R_{\text{max}} = 10$ km and $R_0 = 2$ km !

c) What is the energy density at the maximum mass $M = M_{\text{max}}$?

Discuss the above results in the context of modern constraints for mass and compactness of compact stars !

For details on this task you may consult, e.g., the textbook
N.K. Glendenning, Compact Stars, Springer (2000)