Wykład 8

Cykloida, animacje, prosta grafika, odbicia sprężyste i tłumione, metoda wariacyjna, wahadło podwójne

In[1]:=
```
ClearAll["Global`*"]
```
wyczyść wszystko

Cykloida: animacje

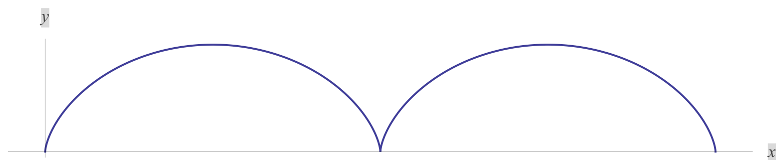
In[2]:=
```
WolframAlpha["Cycloid"]
```
zapytaj WolframAlpha

Assuming "Cycloid" is a plane curve | Use as a word instead

Input interpretation:

cycloid (plane curve)

Plot:



(plotted for t from 0 to 4π)

Alternate names:

brachistochrone curve | Helen of Geometers | tautochrone curve

Equations:

Parametric equations:

$x(t) = a\,(t - \sin(t))$

$y(t) = a\,(1 - \cos(t))$

Cartesian equation:

$$\left|2\pi\left(\left\lceil \frac{1}{2} - \frac{x}{2\,a\,\pi}\right\rceil - 1\right) + \frac{x}{a}\right| = \cos^{-1}\left(1 - \frac{y}{a}\right) - \sqrt{\frac{2\,y}{a} - \frac{y^2}{a^2}}$$

Out[2]=

$\lceil x \rceil$ is the ceiling function »

$|z|$ is the absolute value of z »

$\cos^{-1}(x)$ is the inverse cosine function »

Properties:

parametric | roulette

Basic properties:

Approximate form

Area enclosed:

$A = 3\pi a^2$

Derived curves:

evolute	cycloid
involute	cycloid

Related entities:

Associated people:

Nicholas of Cusa | Charles de Bovelles | Galileo Galilei | Girard Desargues | Marin Mersenne | Gilles Personne de Roberval | René Descartes | Sir Christopher Wren | Christiaan Huygens | Johann Bernoulli | Gottfried Leibniz | Isaac Newton | Jacob Bernoulli | Guillaume François Antoine Marquis de L'Hôpital

WolframAlpha

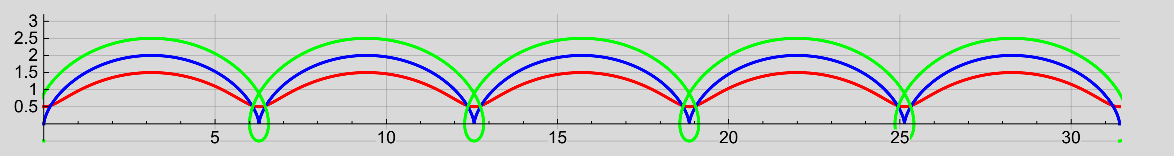
In[3]:=
```
cyklus[t_, r_, c_] := {r (t - c Sin[t]), r (1 - c Cos[t])}
                                 sinus              cosinus

(*
 c -- promień kola rysowanego
 r-- promień koła toczącego się
  rc
*)
```

In[4]:=
```
cyklus[t, r n, 0.5 k]
```

Out[4]= $\{n\,r\,(t - 0.5\,k\sin(t)),\ n\,r\,(1 - 0.5\,k\cos(t))\}$

In[5]:=
```
Show@(ParametricPlot[cyklus[t, First@#, #[[2]]], {t, 0, 10 Pi},
pokaż  wykres parametryczny  pierwszy  pi
      PlotRange → {{0, 10 Pi}, {-0.5, 3.2}}, ImageSize → Large, PlotStyle → Last@#,
      zakres wykresu  pi  rozmiar obrazu  duży  styl grafiki  ostatni
      GridLines → Automatic, Ticks → {Automatic, {0.5, 1, 1.5, 2, 2.5, 3}}] & /@
      linie siatki  automatyczny  znaczki n···  automatyczny
   {{1, 0.5, Red}, {1, 1, Blue}, {1, 1.5, Green}})
           czerwony  niebieski  zielony
```

Out[5]=


In[6]:=
```
Clear[t, r, c]
wyczyść
```

In[7]:=
```
cyklus[t, r, c]
D @@ {%, t}
oblicz pochodną
Print["Srednia prędkość:"]
drukuj
(Integrate @@ {%%, {t, 0, 2 Pi}}) / 2 / Pi
 całka  pi  pi
```

Out[7]= $\{r\,(t - c\sin(t)),\ r\,(1 - c\cos(t))\}$

Out[8]= $\{r\,(1 - c\cos(t)),\ c\,r\sin(t)\}$

Srednia prędkość:

Out[10]= $\{r, 0\}$

Figury geometryczne

In[11]:=
```
? Line
```

Line[{p_1, p_2, …}] represents the line segments joining a sequence for points p_i.

Line[{{p_{11}, p_{12}, …}, {p_{21}, …}, …}] represents a collection of lines. ≫

In[12]:=
```
Graphics[{Red, Thick, Line[{{0, 1}, {0, 0}, {0.5, 0.5}}]}]
```
grafika cz⋯ gruby linia łamana

Out[12]=

In[13]:=
```
?Circle
```

Circle[{x, y}, r] represents a circle of radius r centered at {x, y}.

Circle[{x, y}] gives a circle of radius 1.

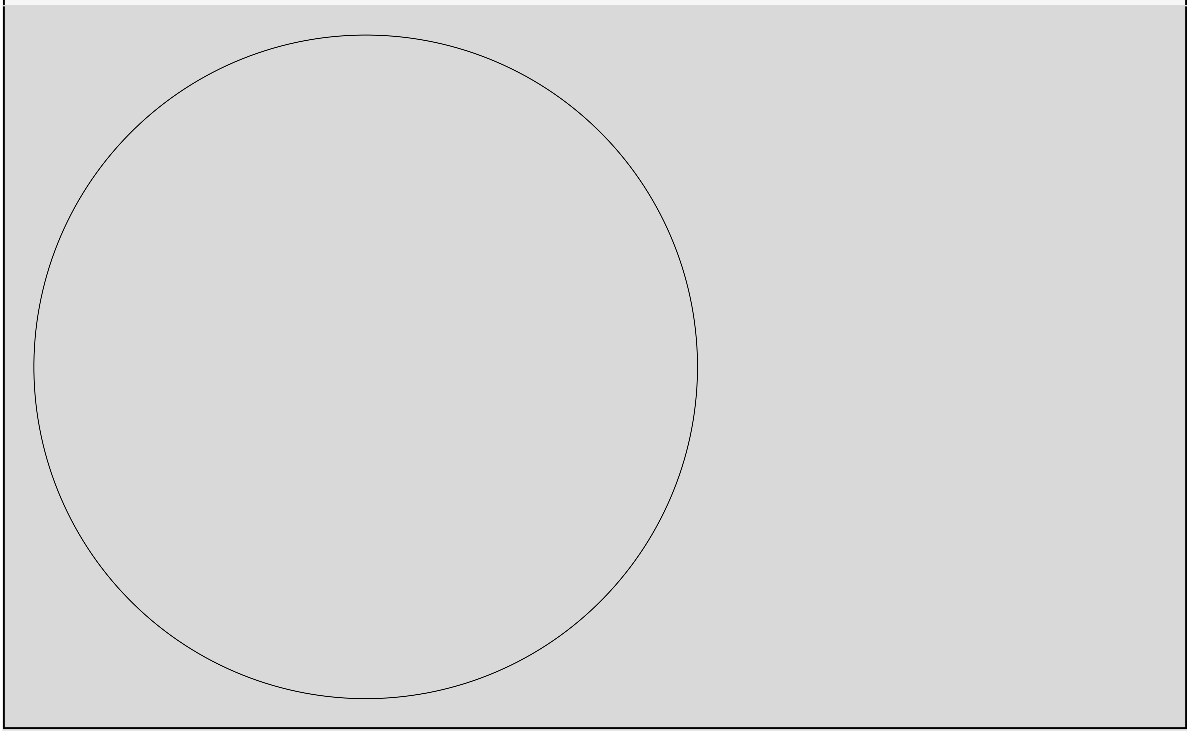
Circle[{x, y}, {r_x, r_y}] gives an axis-aligned ellipse with semi-axes lengths r_x and r_y.

Circle[{x, y}, …, {θ_1, θ_2}] gives a circular or ellipse arc from angle θ_1 to θ_2. >>

In[14]:=
```
Kolo[t_, r_] := Graphics[Circle[{t, r}, r]]
```
grafika okrąg

In[15]:=
```
Kolo[0, 1]
```

Out[15]=



In[16]:=
```
? Point
```

Point[p] is a graphics and geometry primitive that represents a point at p.

Point[{p_1, p_2, …}] represents a collection of points. >>

In[17]:=
```
? Arrow
```

Arrow[{pt_1, pt_2}] is a graphics primitive that represents an arrow from pt_1 to pt_2.

Arrow[{pt_1, pt_2}, s] represents an arrow with its ends set back from pt_1 and pt_2 by a distance s.

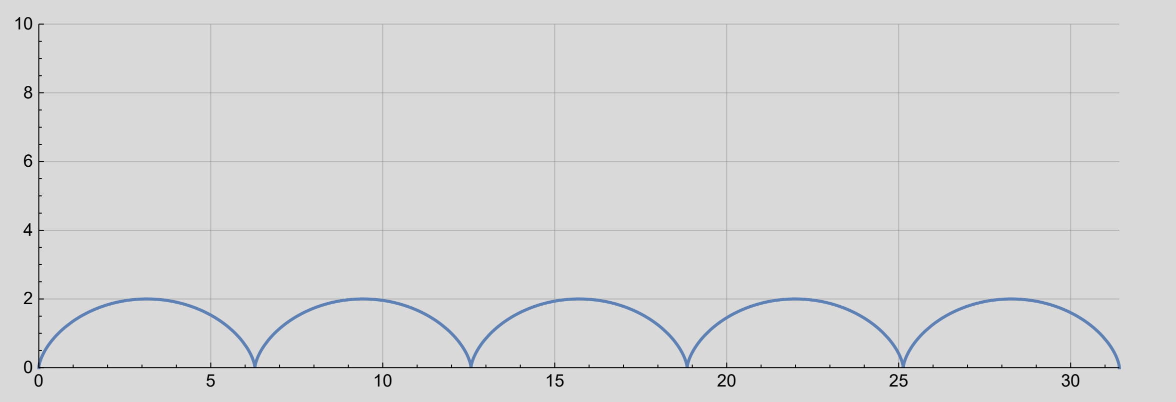
Arrow[{pt_1, pt_2}, {s_1, s_2}] sets back by s_1 from pt_1 and s_2 from pt_2.

Arrow[*curve*, …] represents an arrow following the specified *curve*. >>

Symulujemy ruch cykloidy

In[18]:=
```
ParametricPlot[{(θ - Sin[θ]), (1 - Cos[θ])}, {θ, 0, 10 π},
wykres parametryczny   sinus          cosinus
 PlotRange → {{0, 10 π}, {0, 10}}, ImageSize → Large, GridLines → Automatic]
 zakres wykresu                    rozmiar obrazu duży   linie siatki  automatyczny
```

Out[18]=



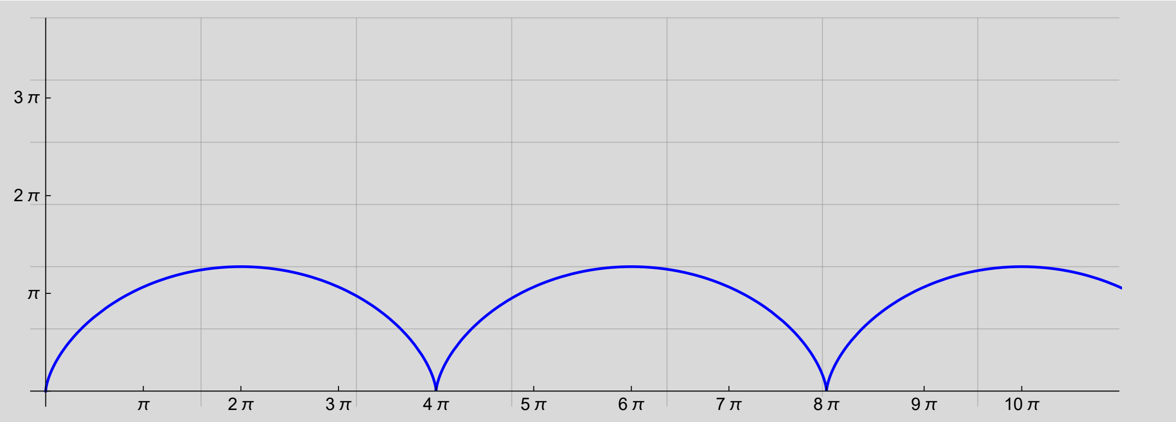
In[19]:=
```
Cykloida[r_, T_, Tmax_, Kolor_] :=
 ParametricPlot[{r (θ - Sin[θ]), r (1 - Cos[θ])}, {θ, 0, T},
 wykres parametryczny    sinus          cosinus
  PlotRange → {{-0.5, 1.1 Tmax}, {-0.5, 6 r }},
  zakres wykresu
  ImageSize → Large, Ticks → {Table[i π, {i, 0, IntegerPart[Tmax / Pi]}]},
              duży   znaczki n·· tabela          część całkowita   pi
  ImageSize → {1000, 400},
  rozmiar obrazu
  PlotStyle → {Thickness[0.0025], Kolor},
  styl grafiki  grubość
  GridLines → Automatic]
  linie siatki  automatyczny
```

In[20]:=
```
? IntegerPart
```

IntegerPart[x] gives the integer part of x. >>

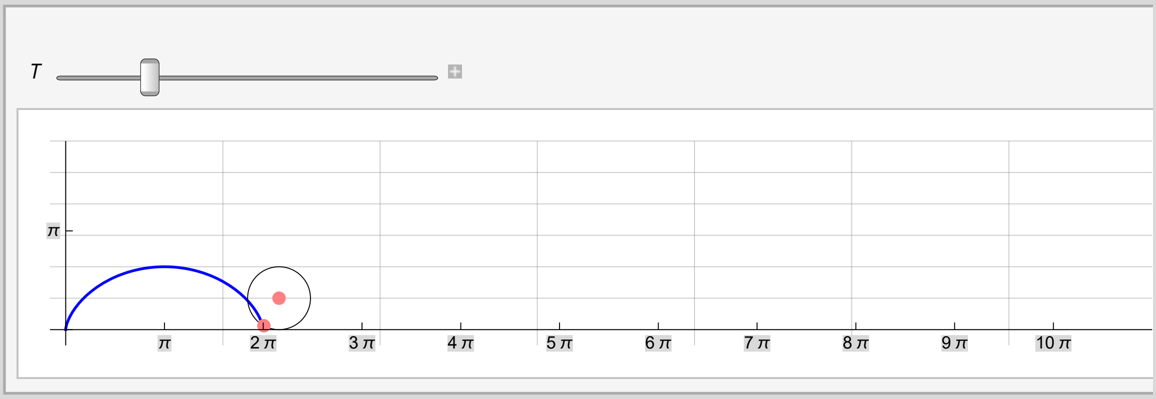
In[21]:=
```
Cykloida[2, 10 Pi, 10 Pi, Blue]
            pi     pi    niebiesk
```

Out[21]=



In[22]:=
```
Manipulate[Show[Cykloida[1, T, 10 Pi, Blue],
zmieniaj   pokaż                    pi   niebieski
  Graphics[Circle[{T, 1}, 1]],
  grafika  okrąg
  Graphics[{Pink, PointSize[Large], Point[{T - Sin[T], 1 - Cos[T]}] }],
  grafika   różowy rozmiar kro··· duży   punkt     sinus       cosinus
  Graphics[{Pink, PointSize[Large], Point[{T, 1}] }]],
  grafika   różowy rozmiar kro··· duży   punkt
 {T, 0.0001, 10 π}]
```

Out[22]=



In[23]:=
```
Odcinek[t_, r_] := Graphics[{Pink, PointSize[Large],
                   grafika    różowy rozmiar kro··· duży
   Point[{r (t - Sin[t]), r (1 - Cos[t])}] , Pink, PointSize[Large],
   punkt      sinus          cosinus          różowy rozmiar kro··· duży
   Point[{t, r}] , Blue, Line[ {{t, r}, {r (t - Sin[t]), r (1 - Cos[t])}}] }]
   punkt          niebi··· linia łamana      sinus          cosinus
```

In[24]:=
```
KoloOdcinek[t_, r_] := Module[{srodek, okrag, T},
                       moduł
  T = t r;
  srodek = {T, r};
  okrag = {r (t - Sin[t]), r (1 - Cos[t])};
              sinus         cosinus
  Graphics[{Circle[srodek, r], Pink, PointSize[Large],
  grafika   okrąg              różowy rozmiar kro··· duży
    Point[okrag] , Pink, PointSize[Large], Point[srodek] ,
    punkt          różowy rozmiar kro··· duży  punkt
    Blue, Line[ {srodek, okrag}] }]
    niebi··· linia łamana
 ]
```

In[25]:=

```
KoloOdcinek[t_, r_, c_] := Module[
                           moduł
  {srodek, okrag, T},
  T = t / r;
  srodek = {t, r};
  okrag = {r (T - c Sin[T]), r (1 - c Cos[T])};
                    sinus             cosinus
  Graphics[{Circle[srodek, r], Pink, PointSize[Large], Point[okrag] ,
  grafika   okrąg               różowy rozmiar kro··· duży  punkt
    Pink, PointSize[Large], Point[srodek] , Blue, Line[ {srodek, okrag}] }]
    różowy rozmiar kro··· duży  punkt            niebi··· linia łamana
 ]
```

In[26]:=

```
? Cykloida
```

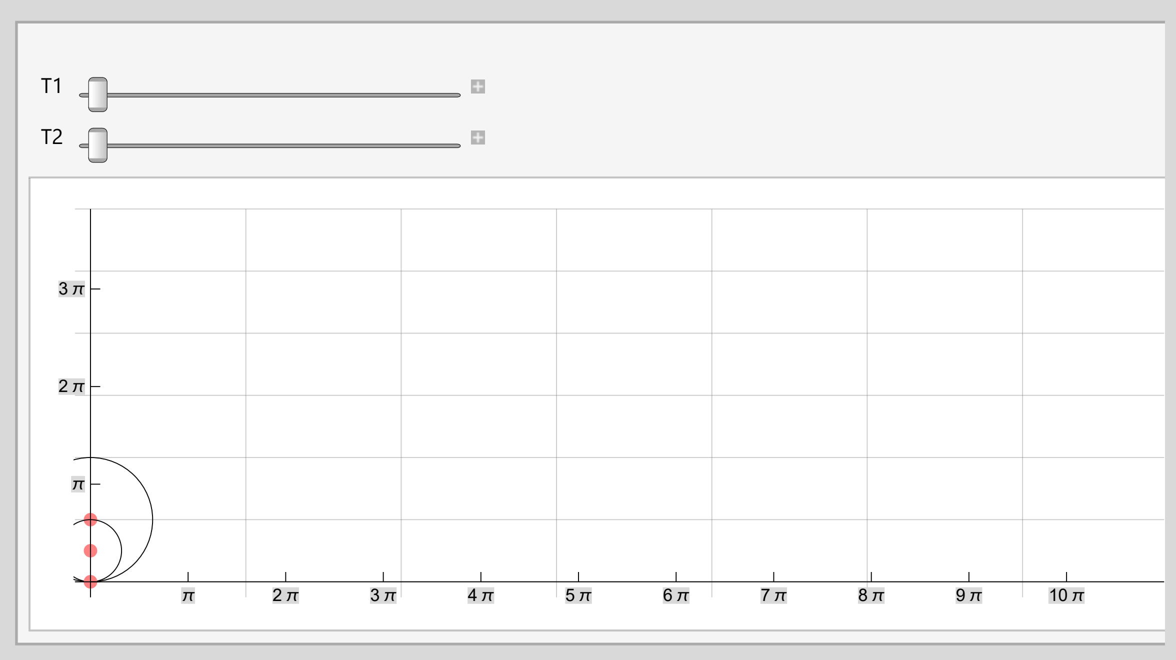
```
Global`Cykloida
```

```
Cykloida[r_, T_, Tmax_, Kolor_] := ParametricPlot[{r (θ - Sin[θ]), r (1 - Cos[θ])},
  {θ, 0, T}, PlotRange → {{-0.5, 1.1 Tmax}, {-0.5, 6 r}}, ImageSize → Large,
  Ticks → {Table[i π, {i, 0, IntegerPart[Tmax/π]}]}, ImageSize → {1000, 400},
  PlotStyle → {Thickness[0.0025], Kolor}, GridLines → Automatic]
```

In[27]:=

```
P1 = Manipulate[
     zmieniaj
  Show[
  pokaż
   Cykloida[2, T1, 10 Pi, Red], KoloOdcinek[T1, 2],
                      pi  czerwony
   Cykloida[1, T2, 10 Pi, Blue], KoloOdcinek[T2, 1]],
                      pi  niebieski
  {T1, 0.0001, 10 π}, {T2, 0.0001, 10 π}]
```

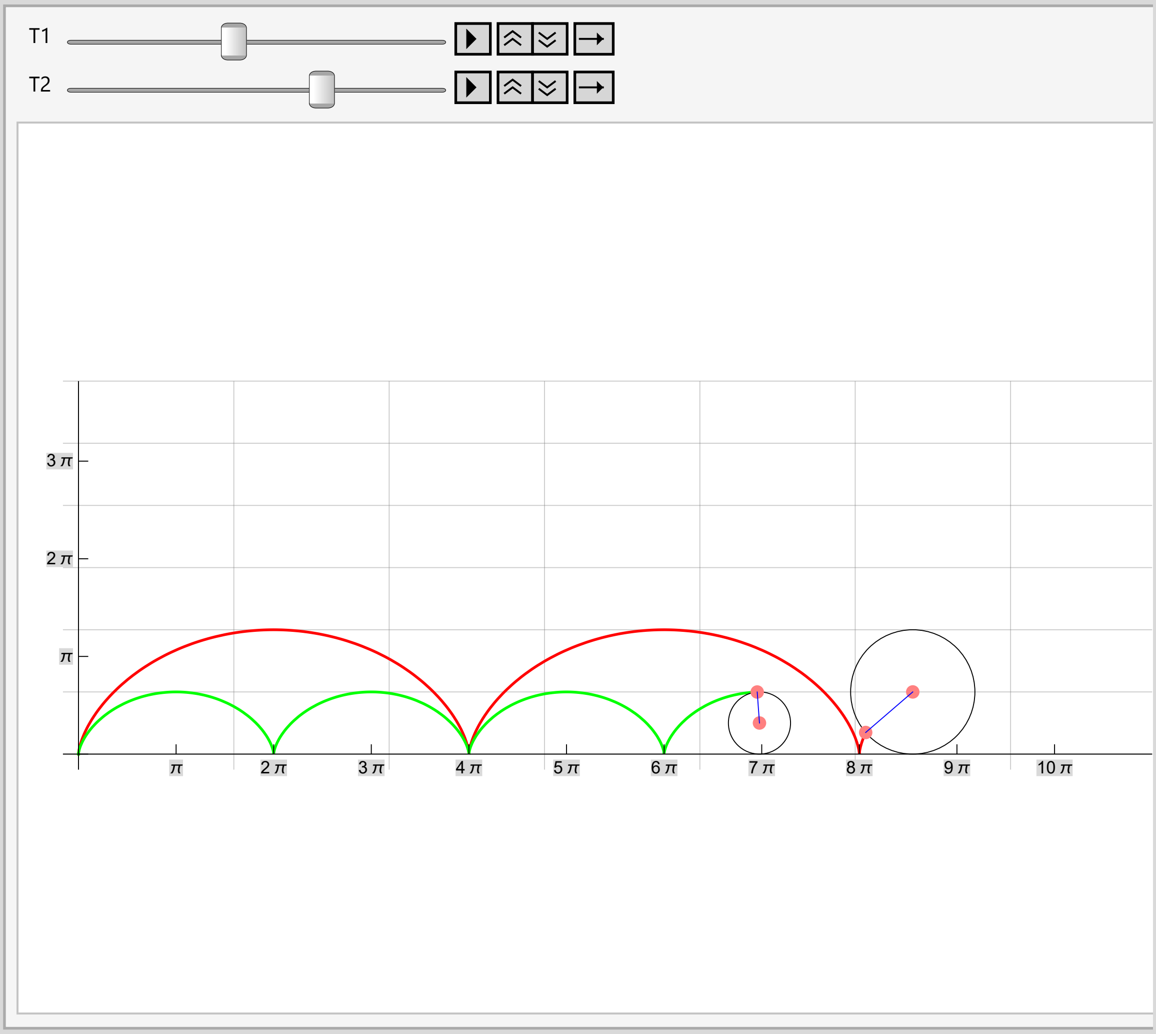
Out[27]=



In[28]:=
```
CalaCykloida[r_, T_, Tmax_, kolor_] :=
 {Cykloida[r, T, 10 Pi, kolor], KoloOdcinek[T, r]}
                  pi
```

In[29]:=
```
Animate[Show[CalaCykloida[2, T1, 10 Pi, Red], CalaCykloida[1, T2, 10 Pi, Green]],
animuj  pokaż                   pi  czerwony                  pi  zielony
 {T1, 0.0001, 10 π}, {T2, 0.0001, 10 π}]
```

Out[29]=

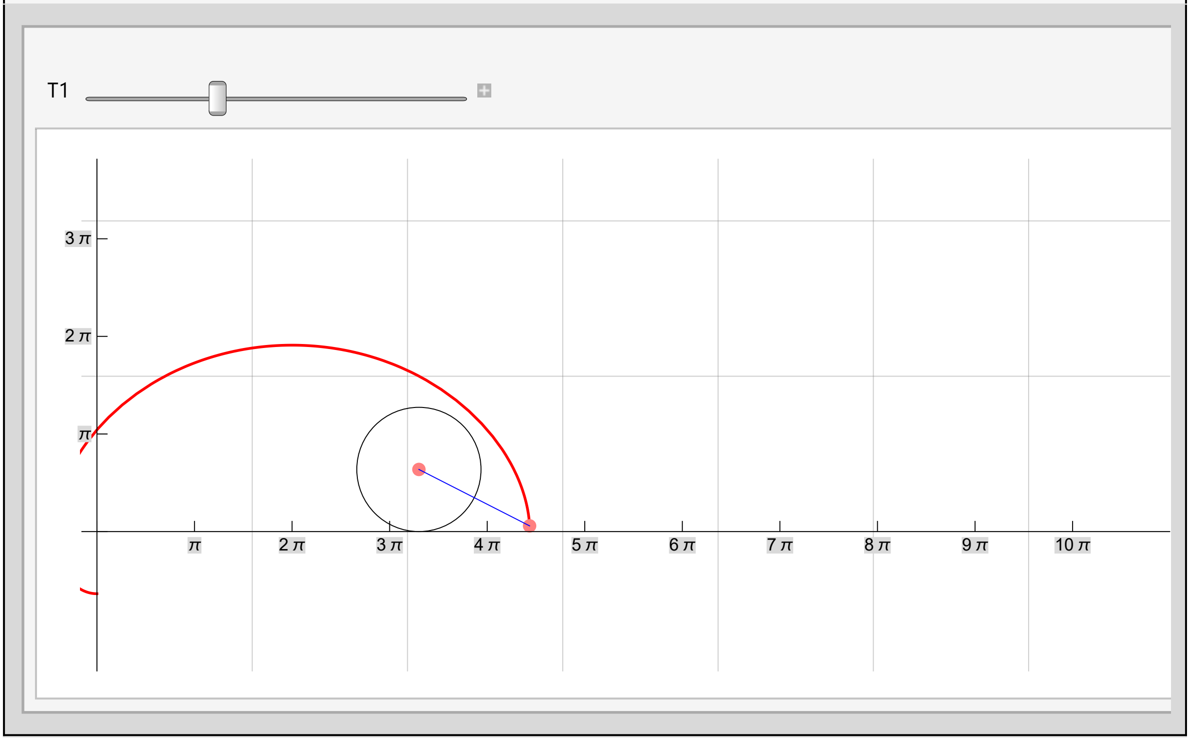


In[30]:=
```
Cykloida[r_, T_, Tmax_, c_, Kolor_] :=
 ParametricPlot[{r (θ / r - c Sin[θ / r]), r (1 - c Cos[θ / r])}, {θ, 0, T},
 wykres parametryczny        sinus               cosinus
  PlotRange → {{-0.5, 1.1 Tmax}, {0 - 2 c - 0.5, 6 r }}, ImageSize → Large,
  zakres wykresu                                      rozmiar obrazu  duży
  Ticks → {Table[i π, {i, 0, IntegerPart[Tmax / Pi]}]}, ImageSize → {1000, 400},
  znaczki n··  tabela      część całkowita     pi       rozmiar obrazu
  PlotStyle → {Thickness[0.0025], Kolor}, GridLines → Automatic]
  styl grafiki  grubość                  linie siatki  automatyczny
```

In[31]:=

```
Manipulate[
zmieniaj
 Show[Cykloida[2, T1, 10 Pi, 2, Red], KoloOdcinek[T1, 2, 2]], {T1, 0.0001, 10 π}]
 pokaż                   pi      czerwony
```

Out[31]=



In[32]:=

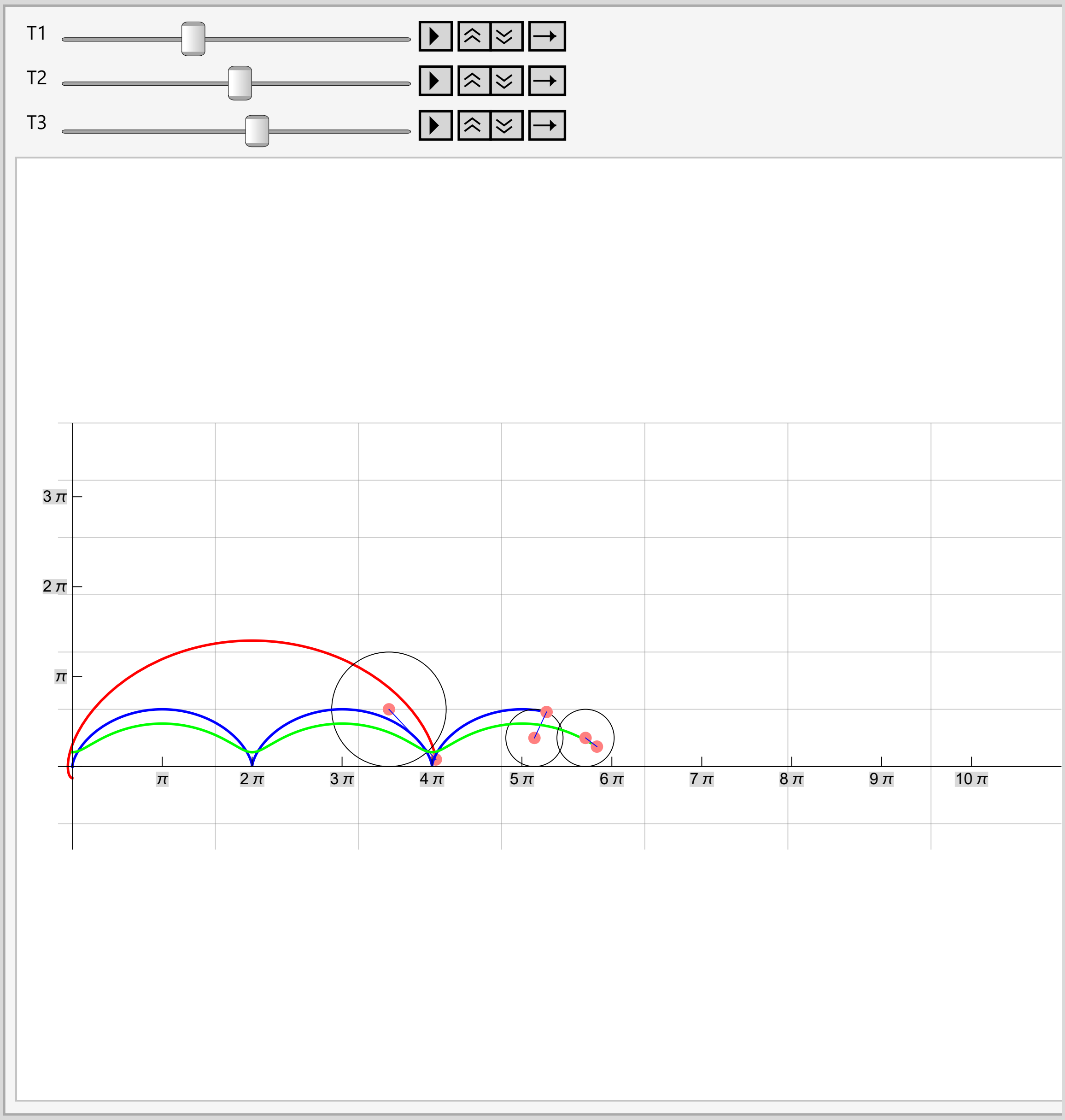
```
CalaCykloida2[r_, T_, Tmax_, c_, Kolor_] :=
 {Cykloida[r, T, 10 Pi, c, Kolor], KoloOdcinek[T, r, c]}
                  pi
```

In[33]:=

```
Animate[Show[CalaCykloida2[2, T1, 10 Pi, 1.2, Red],
  CalaCykloida2[1, T2, 10 Pi, 1, Blue], CalaCykloida2[1, T3, 10 Pi, 0.5, Green]],
 {T1, 0.0001, 10 π}, {T2, 0.0001, 10 π}, {T3, 0.0001, 10 π}]
```

animuj pokaż pi czerwony pi niebieski pi zielony

Out[33]=



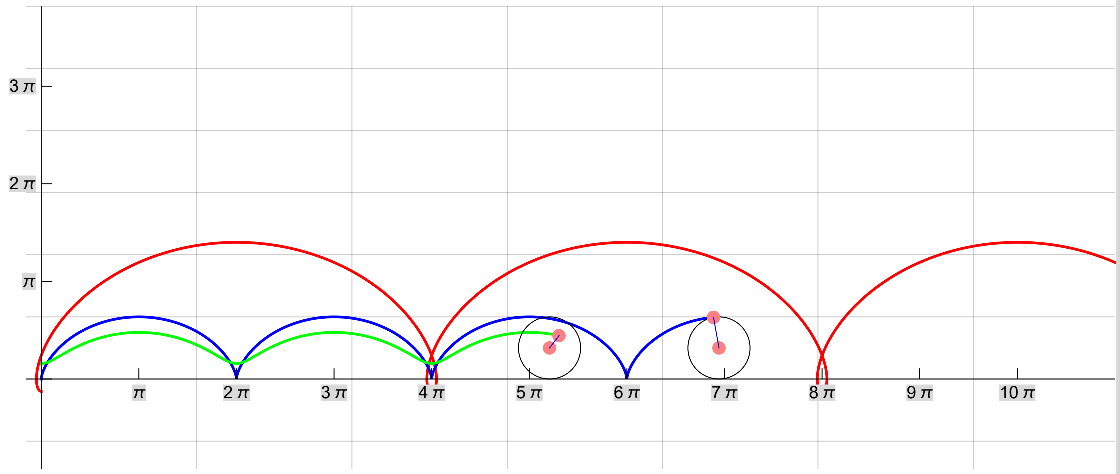
In[34]:=

```
Animate[
 Show[CalaCykloida2[2, T1, 10 Pi, 1.2, Red], CalaCykloida2[1, T1/3, 10 Pi, 1, Blue],
  CalaCykloida2[1, T1/4, 10 Pi, 0.5, Green]], {T1, 0.0001, 35 π}]
```

animuj, pokaż, pi, czerwony, pi, niebieski, pi, zielony

Out[34]=

T1



In[35]:=

```
Export["cykloida.avi", %]
```

eksportuj

Out[35]= cykloida.avi

Spadek swobodny z odbiciami i stratą energii oraz oporem powietrza

Zadanie wygodnie jest rozwiązać korzystając z funkcji NDSolve i opcji WhenEvent.

Rozważmy ciało o masie m=1 spadające swobodnie z wysokości h=5 i odbijające się sprężyście

In[36]:=
```
g =.
g = 9.81
```

Out[37]= 9.81

In[38]:=
```
row0 = y''[t] == -g
```

Out[38]= $y''(t) = -9.81$

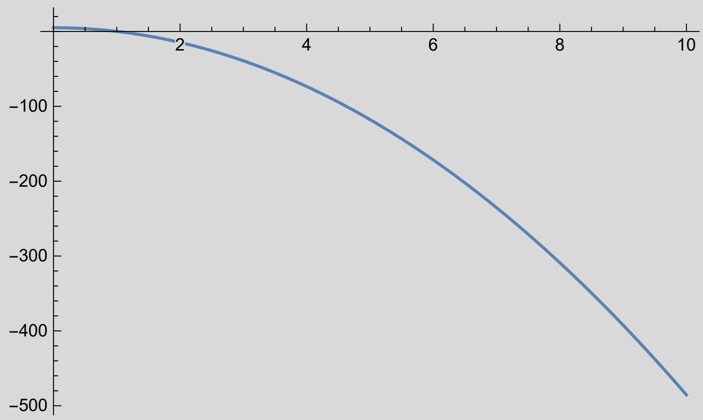
Warunki początkowe y[0]=h=5, y'[0]=0, na razie nie zakładamy, że podłoże jest w y[0]=0

In[39]:=
```
roz0 = NDSolve[{row0, y'[0] == 0, y[0] == 5}, y, {t, 0, 10}]
```
rozwiąż numerycznie równanie różniczkowe

Out[39]= {{y → InterpolatingFunction[Domain: {{0., 10.}} Output: scalar]}}

In[40]:=
```
Plot[y[t] /. roz0, {t, 0, 10}]
```
wykres

Out[40]=



zakładamy, że podłoże jest wysokości y[0]=0 i ciało sprężyście się odbija

In[41]:=
```
? WhenEvent
```

WhenEvent[*event*, *action*] specifies an *action* that occurs when the *event* triggers it for equations in NDSolve and related functions. >>

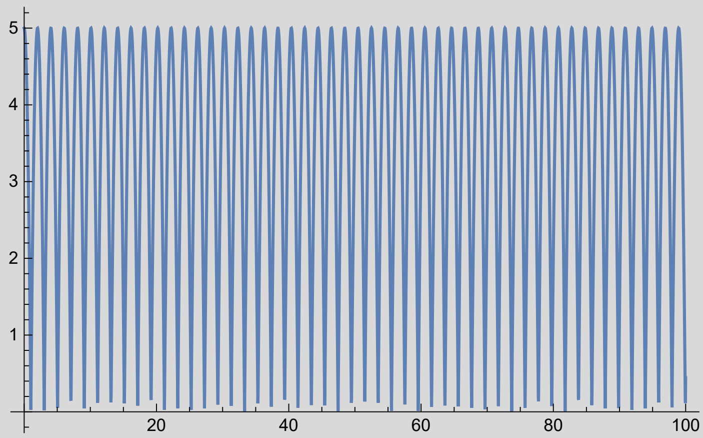
In[42]:=

```
roz1 = NDSolve[
  {row0, y'[0] == 0, y[0] == 5, WhenEvent[y[t] == 0, y'[t] → -y'[t]]}, y, {t, 0, 100}]
Plot[y[t] /. roz1, {t, 0, 100}]
```

rozwiąż numerycznie równanie różniczkowe
akcja spowodowana przez wydarzenia
wykres

Out[42]= {{y → InterpolatingFunction[Domain: {{0., 100.}} Output: scalar]}}

Out[43]=



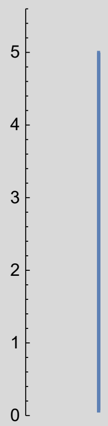
Symulacja

In[44]:=

```
ParametricPlot[{1, y[t]} /. roz1, {t, 0, 10},
 PlotRange → {{-0.5, 2}, {0, 5.6}}, AxesOrigin → {0, -0.5}, ImageSize → Small]
```

wykres parametryczny
zakres wykresu
punkt przecięcia osi
rozmiar obrazu
mały

Out[44]=



In[45]:=

```
? Disk
```

Disk[{x, y}, r] represents a disk of radius r centered at {x, y}.

Disk[{x, y}] gives a disk of radius 1.

Disk[{x, y}, {r_x, r_y}] gives an axis-aligned elliptical disk with semiaxes lengths r_x and r_y.

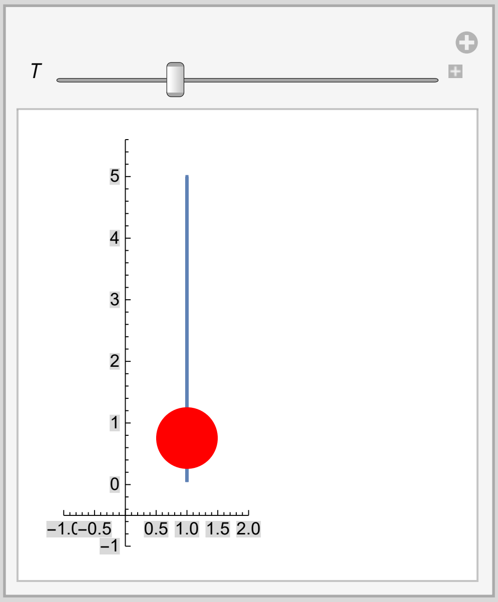
Disk[{x, y}, …, {θ_1, θ_2}] gives a sector of a disk from angle θ_1 to θ_2. ≫

In[46]:=
```
symuluj0[var_, roz_, T_] := Graphics[{Red, Disk[{var, y[T]}, 0.5] /. roz}]
                            grafika    cz⋯  koło
```

In[47]:=
```
symuluj[roz_] := Manipulate[
                 zmieniaj
  Show[
  pokaż
   ParametricPlot[{1, y[t]} /. roz, {t, 0, 10}, PlotRange → {{-1, 2}, {-1, 5.6}},
   wykres parametryczny                          zakres wykresu
    AxesOrigin → {0, -0.5}, ImageSize → Small],
    punkt przecięcia osi    rozmiar obrazu mały
   symuluj0[1, roz, T]], {T, 0.0001, 100, 0.1}]
```

In[48]:=
```
symuluj[roz1]
```

Out[48]=



Opór powietrza proporcjonalny do prędkości

In[49]:=
```
row1 = y''[t] + g + 0.1 y'[t] == 0
```

Out[49]= $y''(t) + 0.1\, y'(t) + 9.81 = 0$

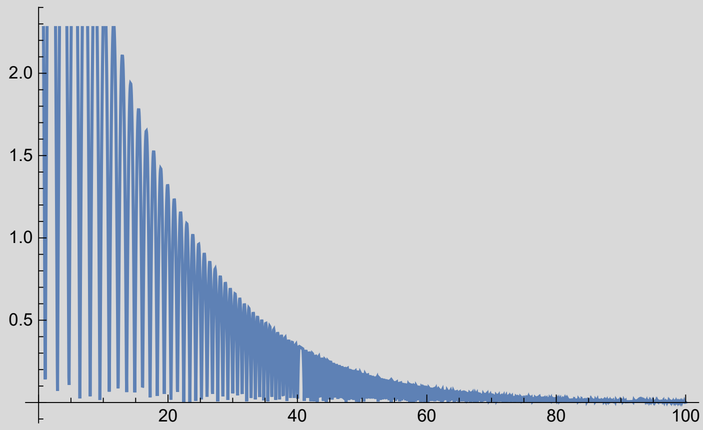
In[50]:=

```
roz2 = NDSolve[
        rozwiąż numerycznie równanie różniczkowe
  {row1, y'[0] == 0, y[0] == 5, WhenEvent[y[t] == 0, y'[t] → -y'[t]]}, y, {t, 0, 100}]
                                akcja spowodowana przez wydarzenia
Plot[y[t] /. roz2, {t, 0, 100}]
wykres
```

Out[50]=

{{y → InterpolatingFunction[Domain: {{0., 100.}} Output: scalar]}}

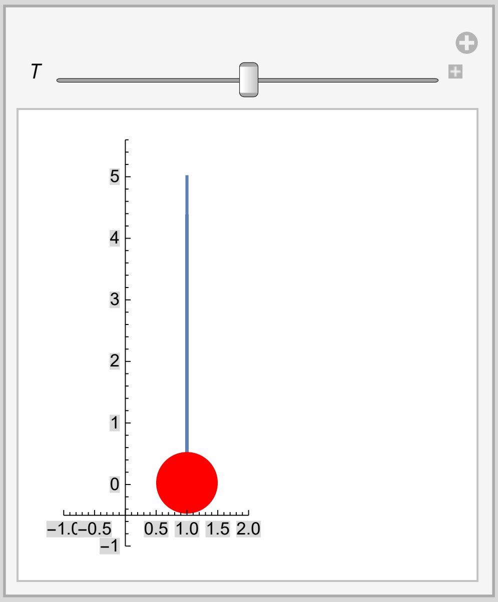
Out[51]=



In[52]:=

```
symuluj[roz2]
```

Out[52]=



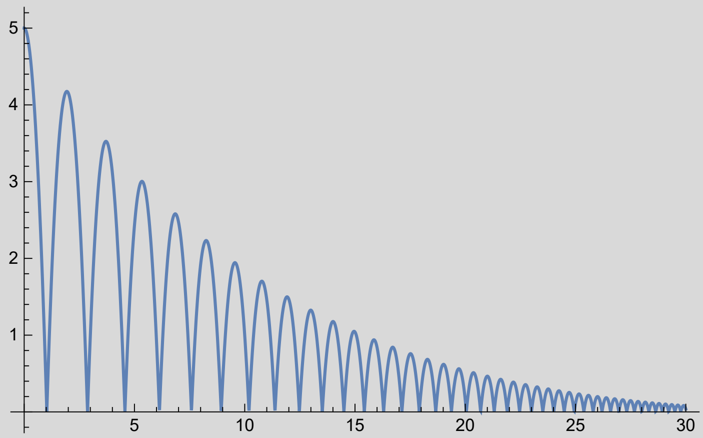
Opór powietrza proporcjonalny do prędkości oraz strata energii 5% energii kinetycznej przy każdym odbiciu

In[53]:=

```
roz3 = NDSolve[{row1, y'[0] == 0, y[0] == 5,
        rozwiąż numerycznie równanie różniczkowe
    WhenEvent[y[t] == 0, y'[t] → -Sqrt[0.95] y'[t]]}, y, {t, 0, 30}]
    akcja spowodowana przez wydarzenia   pierwiastek kwadratowy
Plot[y[t] /. roz3, {t, 0, 30}]
wykres
```

Out[53]= {{y → InterpolatingFunction[Domain: {{0., 30.}} Output: scalar]}}

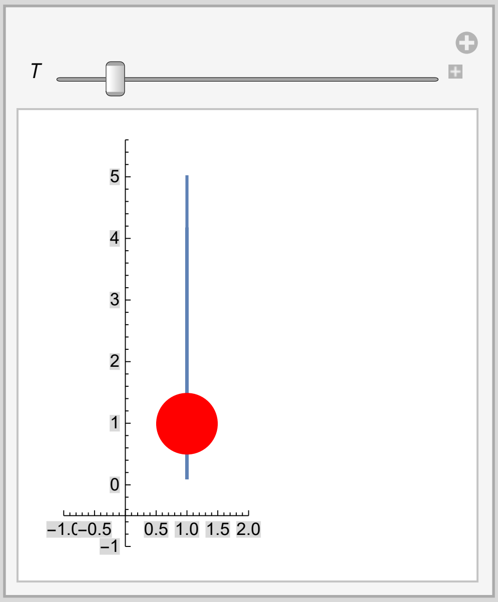
Out[54]=



In[55]:=

```
symuluj[roz3]
```

Out[55]=

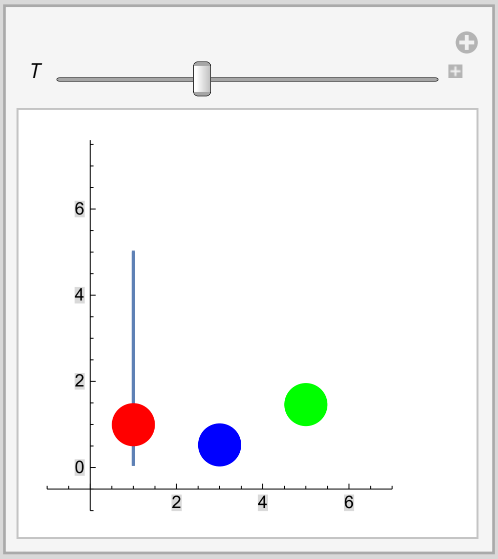


InterpolatingFunction: Input value {32.7001} lies outside the range of data in the interpolating function. Extrapolation will be used.

In[56]:=

```
Manipulate[
zmieniaj
 Show[
 pokaż
  ParametricPlot[{1, y[t]} /. roz1, {t, 0, 10}, PlotRange → {{-1, 7}, {-1, 7.6}},
  wykres parametryczny                                zakres wykresu
   AxesOrigin → {0, -0.5}, ImageSize → Small],
   punkt przecięcia osi      rozmiar obrazu mały
  symuluj0[1, roz1, T], symuluj0[3, roz2, T] /. Red → Blue,
                                               czer⋯  niebieski
  symuluj0[5, roz3, T] /. Red → Green], {T, 0.0001, 30, 0.1}]
                          czer⋯  zielony
```

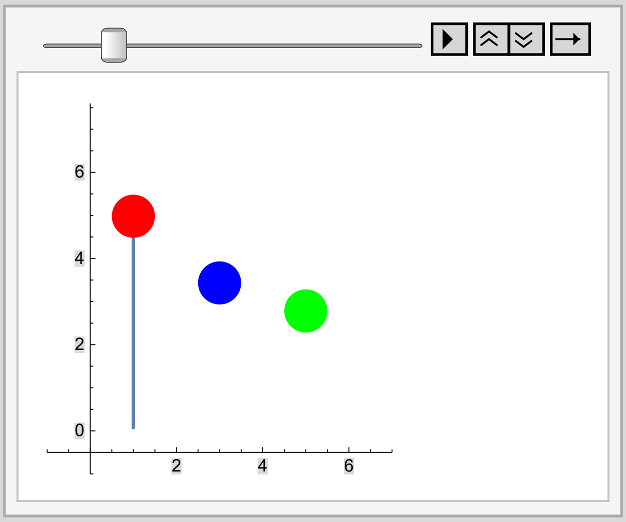
Out[56]=



In[57]:=

```
Table[
tabela
  Show[
  pokaż
   ParametricPlot[{1, y[t]} /. roz1, {t, 0, 10},
   wykres parametryczny
    PlotRange → {{-1, 7}, {-1, 7.6}}, AxesOrigin → {0, -0.5}, ImageSize → Small],
    zakres wykresu                     punkt przecięcia osi           rozmiar obrazu  mały
   symuluj0[1, roz1, T], symuluj0[3, roz2, T] /. Red → Blue,
                                                 czer···  niebieski
   symuluj0[5, roz3, T] /. Red → Green], {T, 0.0001, 30, 0.1}];
                           czer···  zielony
ListAnimate[%, DefaultDuration → 20]
animuj liste   domyślny czas trwania
Export["pilki.avi", %]
eksportuj
```

Out[58]=



Out[59]= pilki.avi

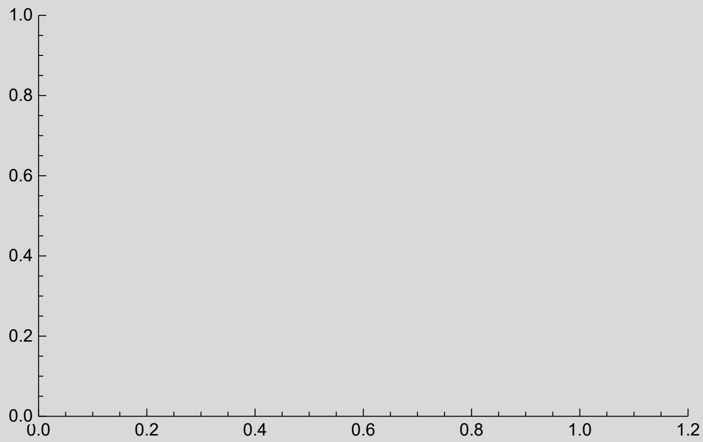
Metoda Wariacyjna

Najkrótsza droga łącząca dwa punkty

Rozważmy dwa punkty w przestrzeni

In[60]:=
```
wyk1 = Plot[, {x, 0, 2}, PlotRange → {{0, 1.2}, {0, 1}}]
        wykres               zakres wykresu
```

Out[60]=



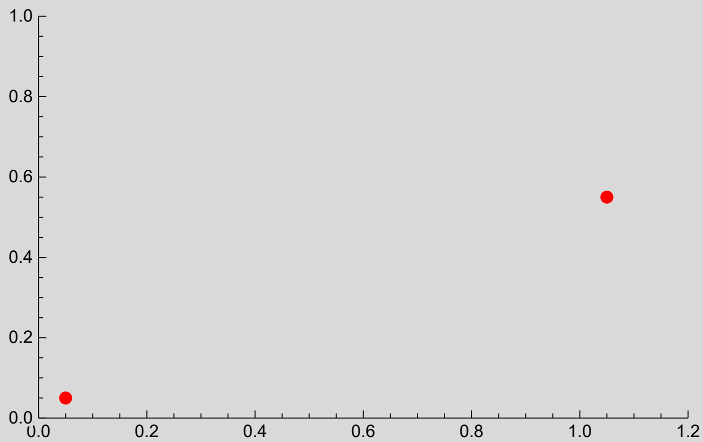
Tworzymy funkcję rysującą czerwony punkt o zadanym rozmiarze

In[61]:=
```
p = {0.05, 0.05}
points =.
points[pp_] := {Graphics[{Red, PointSize → 0.02, Point[pp]}],
                 grafika    cz···  rozmiar kropki   punkt
    Graphics[{Red, PointSize → 0.02, Point[{pp[[1]] + 1, pp[[2]] + 0.5}]}]};
    grafika    cz···  rozmiar kropki   punkt
```

Out[61]= {0.05, 0.05}

In[64]:=
```
Show[wyk1, points[p]]
pokaż
```

Out[64]=



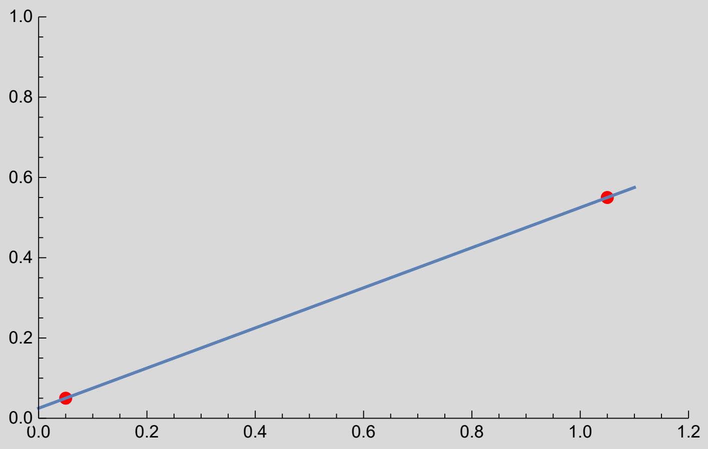
Szukamy równania prostej przechodzącej przez dwa punkty:

In[65]:=
```
sol1 = Solve[{First@p A + B == Last@p, (First@p + 1) A + B == Last@p + 0.5}, {A, B}]
       rozwiąż·· pierwszy        ostatni  pierwszy                ostatni
```

Out[65]= $\{\{A \to 0.5, B \to 0.025\}\}$

In[66]:=
```
Show[wyk1, points[p], Plot[A x + B /. sol1, {x, 0, 1.1}]]
pokaż                     wykres
```

Out[66]=



In[67]:=
```
Clear[MakePoly]
wyczyść
MakePoly[] := Module[{sto},
              moduł
   sto = RandomInteger[{1, 20}];
         losowa liczba całkowita
   (Plus @@ Array[RandomReal[{-3, 3}] x^# &, sto, 0]) (x - p[[1]]) (x - p[[1]] - 1)
    suma     tablic··· losowa liczba rzeczywista
  ];
```

In[69]:=
```
MakePoly[]
```

Out[69]= $(x-1.05)(x-0.05)\left(2.66015 x^4+0.302505 x^3-0.422215 x^2-1.70485 x-1.64686\right)$

In[70]:=
```
Clear[funDowolna]
wyczyść
funDowolna[x_] := ((A x + B) /. Flatten[sol1]) + MakePoly[]
                                spłaszcz
```

In[72]:=
```
funDowolna[x]
```

Out[72]= $(x-1.05)(x-0.05)\left(1.52216 x^6-1.77197 x^5-2.34657 x^4+0.500805 x^3+1.86722 x^2+0.708647 x+0.786575\right)+0.5 x+0.025$

In[73]:=
```
funkcje = Array[funDowolna[x] &, 50]
          tablica wielowymiarowa
```

Out[73]=

$\{0.5 x+(x-1.05)(x-0.05)\left(-0.822911 x^3+2.72501 x^2-0.314782 x-0.473995\right)+0.025,$

$0.5 x+(x-1.05)(x-0.05)\left(-1.20764 x^9-0.714344 x^8+0.606161 x^7+0.139027 x^6+2.95537 x^5+1.77252 x^4-0.958743 x^3-2.30104 x^2-1.09929 x-1.82867\right)+0.025,$

$0.5 x+(x-1.05)(x-0.05)\left(1.31534 x^{17}-2.76777 x^{16}+2.60252 x^{15}+0.607928 x^{14}+2.82421 x^{13}-2.74919 x^{12}-0.269023 x^{11}+2.20306 x^{10}+0.484819 x^9-0.314605 x^8-0.105058 x^7-2.72136 x^6+1.31679 x^5-2.9107 x^4-2.33094 x^3+1.68301 x^2-1.50701 x-0.861258\right)+0.025,$

$0.5\,x+(x-1.05)\,(x-0.05)\,\big(2.12667\,x^{14}+0.941325\,x^{13}+0.746511\,x^{12}-2.24466\,x^{11}+$
$2.14132\,x^{10}-0.0763908\,x^{9}+2.11044\,x^{8}-0.446847\,x^{7}+1.14024\,x^{6}-0.141331\,x^{5}-$
$2.01264\,x^{4}-0.52122\,x^{3}-0.827959\,x^{2}+2.69902\,x+1.77229\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(1.47153\,x^{7}-2.91017\,x^{6}+2.73532\,x^{5}+0.413779\,x^{4}-$
$2.38376\,x^{3}-1.13192\,x^{2}-1.84408\,x+2.89224\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-0.613132\,x^{7}-0.938088\,x^{6}+2.88084\,x^{5}-1.80717\,x^{4}+$
$0.214826\,x^{3}-0.308929\,x^{2}-2.77368\,x+0.286196\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-1.19247\,x^{4}-1.2616\,x^{3}-0.639701\,x^{2}-2.7778\,x-0.265854\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-1.51274\,x^{11}+1.03762\,x^{10}-2.57799\,x^{9}-1.72302\,x^{8}+2.50352\,x^{7}+$
$0.227045\,x^{6}-2.18927\,x^{5}-0.275734\,x^{4}+2.68114\,x^{3}-1.55548\,x^{2}-1.89243\,x+1.96384\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-1.66876\,x^{19}-0.243818\,x^{18}-0.694933\,x^{17}+1.79644\,x^{16}+$
$0.737636\,x^{15}-1.06033\,x^{14}+0.294042\,x^{13}-0.243564\,x^{12}+2.21014\,x^{11}-$
$2.15761\,x^{10}-2.63759\,x^{9}-0.608382\,x^{8}-1.78483\,x^{7}+1.38796\,x^{6}-2.10788\,x^{5}+$
$0.98156\,x^{4}+2.73834\,x^{3}-0.287748\,x^{2}+2.25028\,x+0.857641\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-2.06387\,x^{4}+1.25476\,x^{3}-2.52337\,x^{2}-0.814006\,x+1.18396\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-2.03461\,x^{17}+0.0820451\,x^{16}+1.46457\,x^{15}+0.686751\,x^{14}+1.22816\,x^{13}+$
$1.94282\,x^{12}+1.87962\,x^{11}-1.18014\,x^{10}-0.932258\,x^{9}+1.77097\,x^{8}-0.749833\,x^{7}+0.59033\,x^{6}-$
$2.47741\,x^{5}+0.939084\,x^{4}+1.99533\,x^{3}+2.60687\,x^{2}+0.306769\,x+0.168892\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(2.40095\,x^{5}-1.19394\,x^{4}-2.9661\,x^{3}-0.100644\,x^{2}+1.39384\,x+0.808972\big)+$
$0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-0.318766\,x^{18}+2.78942\,x^{17}-1.35235\,x^{16}+0.957785\,x^{15}+2.53601\,x^{14}-$
$0.916549\,x^{13}+0.251598\,x^{12}+0.670278\,x^{11}+0.72371\,x^{10}+2.75078\,x^{9}-1.31272\,x^{8}+1.86675\,x^{7}-$
$1.95556\,x^{6}+0.174783\,x^{5}-1.09282\,x^{4}-0.614515\,x^{3}-0.39889\,x^{2}+1.70844\,x-1.87421\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-1.77221\,x^{2}-0.249018\,x+2.7477\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)$
$\big(1.99004\,x^{6}-1.75982\,x^{5}+2.36994\,x^{4}+1.89205\,x^{3}-0.825058\,x^{2}-2.12706\,x+0.203474\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-0.110352\,x^{15}+0.755807\,x^{14}-2.42064\,x^{13}+0.387537\,x^{12}+$
$0.412159\,x^{11}-2.67956\,x^{10}+1.28972\,x^{9}+0.760434\,x^{8}-0.260894\,x^{7}-1.49958\,x^{6}+$
$0.592633\,x^{5}-1.40956\,x^{4}+2.58987\,x^{3}-0.573869\,x^{2}-2.0244\,x-0.790692\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(0.94724\,x^{15}+1.35223\,x^{14}-0.089935\,x^{13}+2.51402\,x^{12}-$
$2.08233\,x^{11}+0.499236\,x^{10}+0.463356\,x^{9}+0.858774\,x^{8}-1.21938\,x^{7}+1.62991\,x^{6}-$
$1.83001\,x^{5}+1.29452\,x^{4}+1.21376\,x^{3}-1.59304\,x^{2}+2.8777\,x-0.586055\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-0.294348\,x^{14}-1.56119\,x^{13}-1.50086\,x^{12}+0.595826\,x^{11}-$
$0.363407\,x^{10}-2.50626\,x^{9}+1.39312\,x^{8}+1.51652\,x^{7}+0.907284\,x^{6}-0.514903\,x^{5}-$
$0.220411\,x^{4}+2.38295\,x^{3}+0.788741\,x^{2}-1.23554\,x-2.90411\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-2.8374\,x^{8}-1.86625\,x^{7}+1.21807\,x^{6}+1.16643\,x^{5}-$
$2.67459\,x^{4}-2.83081\,x^{3}-1.91114\,x^{2}+1.43379\,x+2.72318\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-0.349448\,x^{17}+2.95025\,x^{16}-0.0223351\,x^{15}+1.45952\,x^{14}+2.86668\,x^{13}-$
$2.19462\,x^{12}+2.76289\,x^{11}+1.38917\,x^{10}-1.14759\,x^{9}-2.72966\,x^{8}-0.873228\,x^{7}-2.82373\,x^{6}+$
$0.118402\,x^{5}-1.10431\,x^{4}-0.0682041\,x^{3}+0.76666\,x^{2}+1.96836\,x+1.00681\big)+0.025,$
$0.5\,x+(x-1.05)\,(x-0.05)\,\big(-0.384548\,x^{17}+1.68308\,x^{16}+0.433994\,x^{15}+0.530249\,x^{14}+2.57396\,x^{13}-$
$1.66604\,x^{12}-1.95895\,x^{11}-1.83888\,x^{10}-2.65629\,x^{9}+1.36106\,x^{8}+2.57621\,x^{7}+1.8445\,x^{6}-$
$2.27607\,x^{5}+0.271504\,x^{4}+2.84263\,x^{3}-1.35358\,x^{2}-1.83945\,x+1.26092\big)+0.025,\ 0.5\,x+$

$(x - 1.05)(x - 0.05)\left(-2.33317 x^{12} - 1.13483 x^{11} - 0.60848 x^{10} - 2.77114 x^{9} - 0.636168 x^{8} - 2.0009 x^{7} + \right.$
$\left. 2.67735 x^{6} - 0.843113 x^{5} + 0.154077 x^{4} - 0.398737 x^{3} + 1.82915 x^{2} + 2.83873 x + 1.18747\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(-0.808045 x^{14} - 0.480662 x^{13} - 2.10344 x^{12} - 0.491334 x^{11} - \right.$
$0.470553 x^{10} - 0.480545 x^{9} - 0.550335 x^{8} + 1.44527 x^{7} - 1.91437 x^{6} +$
$\left. 2.16276 x^{5} + 1.1497 x^{4} + 1.6409 x^{3} - 1.69207 x^{2} - 0.235505 x + 0.265135\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(-1.42845 x^{5} - 0.530125 x^{4} - 1.68126 x^{3} - 2.66019 x^{2} + 0.121784 x + 0.788415\right) +$
$0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(0.901744 x^{18} + 2.1626 x^{17} - 0.379505 x^{16} + 2.6229 x^{15} - 0.203938 x^{14} - \right.$
$0.234115 x^{13} + 0.294417 x^{12} + 1.85073 x^{11} + 2.71438 x^{10} - 1.89436 x^{9} + 0.728953 x^{8} - 1.35255 x^{7} +$
$\left. 2.05477 x^{6} + 0.330901 x^{5} + 0.880486 x^{4} + 1.75032 x^{3} - 0.0017238 x^{2} + 1.70936 x - 2.29106\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(2.23599 x^{8} - 2.00518 x^{7} + 2.39018 x^{6} - 2.52493 x^{5} - 0.000707513 x^{4} + \right.$
$\left. 2.37389 x^{3} - 2.15524 x^{2} + 2.05049 x - 0.348179\right) + 0.025,\ 0.5 x + (x - 1.05)(x - 0.05)$
$\left(0.0726365 x^{6} - 0.715576 x^{5} + 0.92266 x^{4} - 0.118488 x^{3} - 1.19585 x^{2} + 0.318021 x - 2.7724\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(1.39961 x^{8} + 1.3555 x^{7} - 0.947858 x^{6} - 1.70178 x^{5} - \right.$
$\left. 0.724387 x^{4} + 0.808644 x^{3} - 1.19504 x^{2} - 2.72286 x + 0.83883\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(-2.43793 x^{2} + 2.17322 x + 2.216\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(-0.258402 x^{7} - 2.26804 x^{6} + 0.918187 x^{5} - 2.62315 x^{4} + 1.44063 x^{3} + \right.$
$\left. 1.60702 x^{2} + 2.78308 x + 0.595478\right) + 0.025,\ 1.30306 (x - 1.05)(x - 0.05) + 0.5 x + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(1.2918 x^{17} - 1.34459 x^{16} - 1.20046 x^{15} + 0.0274141 x^{14} - 1.78715 x^{13} + \right.$
$1.7255 x^{12} - 0.38474 x^{11} + 1.90089 x^{10} + 2.20317 x^{9} - 2.27088 x^{8} + 1.27445 x^{7} - 1.12393 x^{6} -$
$\left. 2.4984 x^{5} - 1.41905 x^{4} - 0.369125 x^{3} - 1.58173 x^{2} - 2.03747 x + 2.78153\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(1.04654 x^{17} + 0.143837 x^{16} - 2.27459 x^{15} - 0.827757 x^{14} + 2.96943 x^{13} - \right.$
$1.0999 x^{12} + 1.05901 x^{11} - 0.651063 x^{10} - 1.70803 x^{9} + 1.20505 x^{8} - 2.79686 x^{7} - 1.83439 x^{6} -$
$\left. 1.95251 x^{5} - 0.686661 x^{4} - 2.87625 x^{3} + 1.52585 x^{2} + 0.696671 x - 2.33\right) + 0.025,\ 0.5 x +$
$(x - 1.05)(x - 0.05)\left(-0.601959 x^{19} + 1.782 x^{18} + 1.59252 x^{17} - 2.02868 x^{16} + 0.68647 x^{15} + 1.73045 x^{14} + \right.$
$2.85429 x^{13} + 0.410353 x^{12} + 2.40275 x^{11} + 2.84025 x^{10} + 2.26211 x^{9} - 0.646734 x^{8} + 1.8833 x^{7} -$
$\left. 2.14154 x^{6} + 1.03903 x^{5} + 1.94974 x^{4} + 2.4106 x^{3} + 0.668446 x^{2} + 1.37452 x + 0.843588\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(0.894372 x^{8} - 2.91275 x^{7} + 0.792682 x^{6} - 1.92252 x^{5} - \right.$
$\left. 2.13922 x^{4} - 0.616175 x^{3} - 0.939283 x^{2} + 2.98867 x + 0.195084\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(0.571096 x^{12} - 1.03887 x^{11} + 2.83515 x^{10} + 1.7349 x^{9} + 2.42678 x^{8} - 1.59849 x^{7} + \right.$
$\left. 2.31921 x^{6} - 2.39095 x^{5} + 1.35004 x^{4} - 0.474423 x^{3} + 2.09875 x^{2} - 0.286895 x - 1.3652\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(2.0935 x^{12} + 2.92586 x^{11} + 1.85836 x^{10} + 0.583534 x^{9} + 2.12065 x^{8} + 0.799776 \right.$
$\left. x^{7} + 0.516462 x^{6} - 0.37142 x^{5} + 1.90096 x^{4} + 2.4266 x^{3} + 0.296272 x^{2} + 2.87815 x + 0.489591\right) +$
$0.025,\ 0.5 x + (x - 1.05)(x - 0.05)\left(1.92258 x^{2} - 2.62032 x + 1.57178\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)$
$\left(-0.473863 x^{6} + 2.75376 x^{5} + 2.51513 x^{4} - 0.0290147 x^{3} - 1.21175 x^{2} - 0.871727 x - 2.34989\right) + 0.025,$
$2.8811 (x - 1.05)(x - 0.05) + 0.5 x + 0.025,\ 0.5 x + (x - 1.05)(x - 0.05)$
$\left(0.680407 x^{19} + 1.34876 x^{18} - 0.114408 x^{17} - 0.125799 x^{16} + 1.83263 x^{15} - 0.753239 x^{14} - 2.2672 x^{13} - \right.$
$2.94148 x^{12} + 0.377752 x^{11} + 0.392484 x^{10} + 0.278649 x^{9} - 2.96751 x^{8} - 1.07405 x^{7} +$
$\left. 0.88401 x^{6} - 2.67021 x^{5} - 0.44331 x^{4} - 1.41837 x^{3} + 0.247719 x^{2} - 0.72048 x - 1.55101\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(0.0294403 x^{6} + 2.95289 x^{5} + 2.23927 x^{4} + 2.6376 x^{3} + \right.$
$\left. 2.54245 x^{2} - 1.55388 x - 2.19999\right) + 0.025,$
$0.5 x + (x - 1.05)(x - 0.05)\left(-1.55616 x^{15} + 0.250674 x^{14} - 1.52467 x^{13} + 0.170444 x^{12} - \right.$

$$2.44457\,x^{11} - 2.08228\,x^{10} + 2.54612\,x^9 - 2.58426\,x^8 - 0.411069\,x^7 - 0.106445\,x^6 + 2.07092\,x^5 + 2.77034\,x^4 + 0.321825\,x^3 + 2.59907\,x^2 - 1.57311\,x - 1.00583) + 0.025,$$

$$-1.21054\,(x - 1.05)\,(x - 0.05) + 0.5\,x + 0.025,\ 0.5\,x + (x - 1.05)\,(x - 0.05)\,(0.563621\,x^9 + 1.50319\,x^8 - 1.43384\,x^7 - 1.42132\,x^6 - 0.805571\,x^5 + 1.96688\,x^4 + 2.56989\,x^3 + 1.02337\,x^2 + 2.24757\,x + 2.91832) + 0.025,$$

$$0.5\,x + (x - 1.05)\,(x - 0.05)\,(1.98401\,x^4 + 0.258728\,x^3 - 1.28709\,x^2 + 1.48405\,x + 0.545265) + 0.025,$$

$$0.5\,x + (x - 1.05)\,(x - 0.05)\,(-0.955865\,x^{14} + 2.62501\,x^{13} + 1.75471\,x^{12} + 1.50658\,x^{11} + 2.48244\,x^{10} - 2.1943\,x^9 - 0.386219\,x^8 - 1.82063\,x^7 + 0.591487\,x^6 - 1.20229\,x^5 - 0.567739\,x^4 - 1.4997\,x^3 - 0.308725\,x^2 - 2.40886\,x + 0.630155) + 0.025,$$

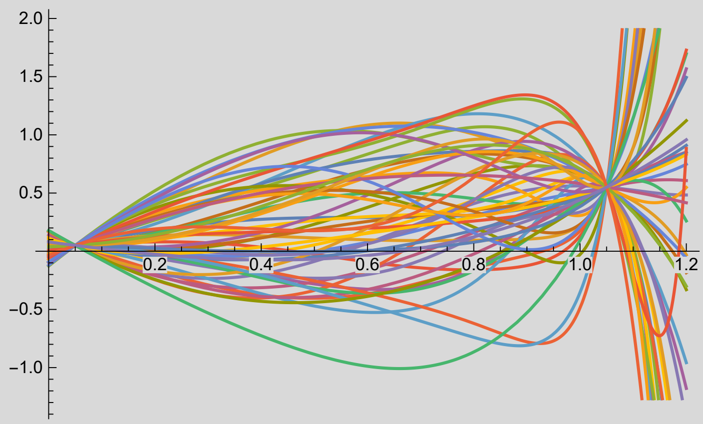
$$0.5\,x + (x - 1.05)\,(x - 0.05)\,(-2.862\,x^2 - 1.24415\,x + 0.249336) + 0.025,$$

$$0.5\,x + (x - 1.05)\,(x - 0.05)\,(2.21469\,x^{19} - 0.823673\,x^{18} - 0.98148\,x^{17} - 1.37911\,x^{16} + 0.040128\,x^{15} - 0.544584\,x^{14} + 0.826666\,x^{13} - 0.899231\,x^{12} + 1.01847\,x^{11} - 1.9173\,x^{10} - 2.07462\,x^9 - 1.06362\,x^8 - 1.76017\,x^7 - 1.43533\,x^6 - 2.5473\,x^5 - 2.38486\,x^4 - 0.580225\,x^3 + 2.20336\,x^2 + 2.64951\,x - 1.13019) + 0.025,$$

$$0.5\,x + (x - 1.05)\,(x - 0.05)\,(0.0837292\,x^2 + 0.633029\,x + 1.04875) + 0.025\}$$

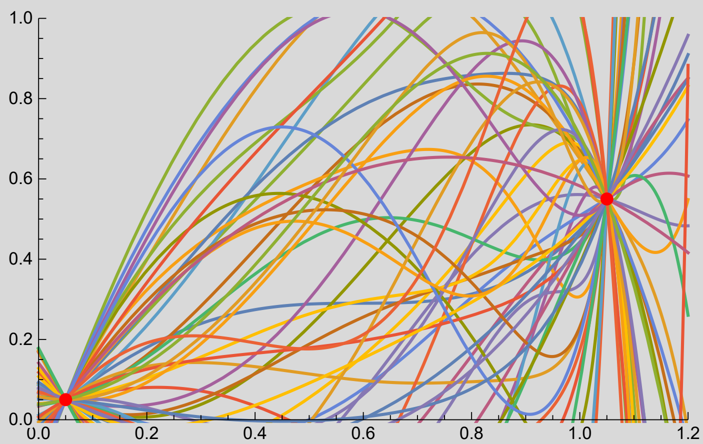
In[74]:=
```
wyk2 = Plot[funkcje, {x, 0, 1.2}]
         wykres
```

Out[74]=



In[75]:=
```
Show[wyk1, wyk2, points[p]]
pokaż
```

Out[75]=



Szukamy najkrótszej drogi

$$S = \int_{(0,0)}^{(1.1)} \sqrt{x^2(t) + y^2(t)}\, dt = \int_{(0,0)}^{(1.1)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

In[76]:=

```
? Min
```

Min[x_1, x_2, …] yields the numerically smallest of the x_i.

Min[{x_1, x_2, …}, {y_1, …}, …] yields the smallest element of any of the lists. ≫

In[77]:=

```
Sqrt[1 + D[#, x]^2] & /@ funkcje;
NIntegrate[#, {x, 0, 1}] & /@ %
Min[%]
Position[%%, %][[1]][[1]]
funkcje[[%]]
Plot[%, {x, 0, 1}, PlotStyle → {Dashed, Thick, Black}]
finale0 = Show[wyk1, wyk2, points[p], %]
```

pierwias··· oblicz pochodną; numeryczne przybliżenie całki; minimum; pozycja; wykres; styl grafiki; linia prz···; gruby; czarny; pokaż

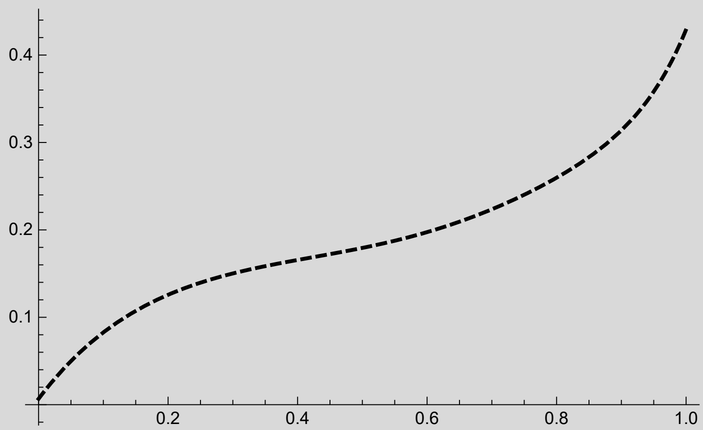
Out[78]= {1.1552, 1.95828, 1.81074, 1.53456, 1.63146, 1.43441, 1.87861, 1.4565, 1.72885, 1.38735, 1.25172, 1.24215, 1.5771, 1.68807, 1.20799, 1.469, 1.12264, 1.91073, 2.17605, 1.39374, 1.14212, 2.13108, 1.22249, 1.58811, 1.79368, 1.11958, 1.95726, 1.49034, 1.76012, 1.66592, 1.27721, 1.87898, 2.01083, 1.61553, 1.45403, 1.48859, 1.76246, 1.20726, 1.97446, 1.84115, 1.99865, 2.0694, 1.51761, 1.32832, 2.57689, 1.19041, 1.69961, 1.52244, 1.79699, 1.27307}

Out[79]= 1.11958

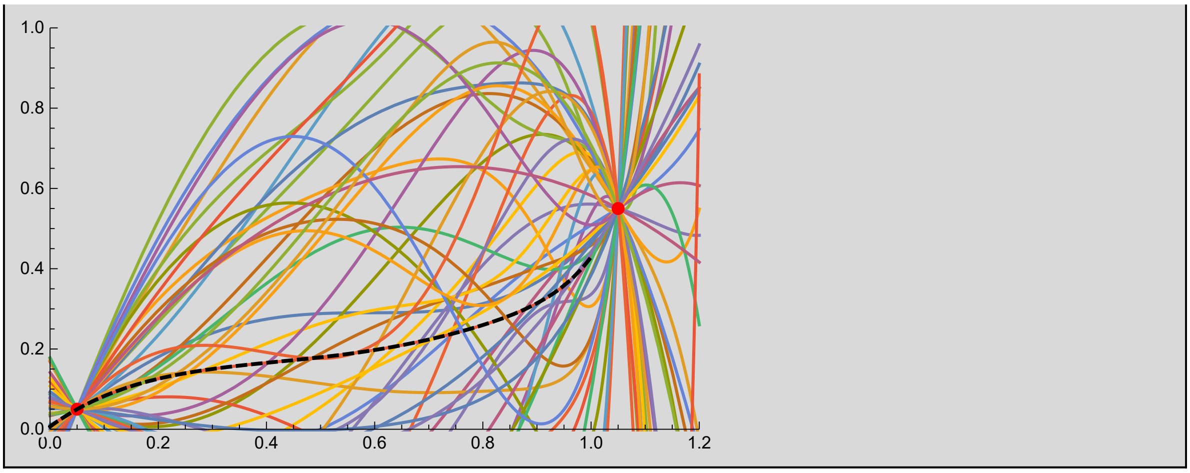
Out[80]= 26

Out[81]= $(x - 1.05)(x - 0.05)\left(2.23599\,x^8 - 2.00518\,x^7 + 2.39018\,x^6 - 2.52493\,x^5 - 0.000707513\,x^4 + 2.37389\,x^3 - 2.15524\,x^2 + 2.05049\,x - 0.348179\right) + 0.5\,x + 0.025$

Out[82]=



Out[83]=



Pakiet do metod wariacyjnych

In[84]:=
```
Needs["VariationalMethods`"]
```
wymaga

In[85]:=
```
<< "VariationalMethods`"
```

In[86]:=
```
? VariationalMethods`*
```

▼ VariationalMethods`

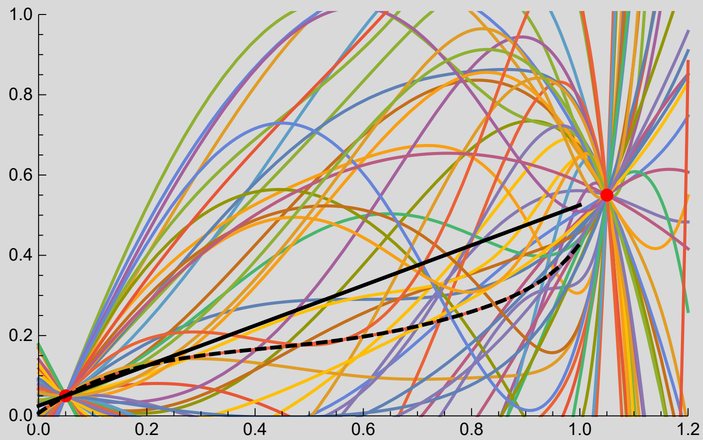
EulerEquations	FirstIntegrals	VariationalBound
FirstIntegral	NVariationalBound	VariationalD

In[87]:=
```
EulerEquations[Sqrt[1 + y'[x]^2], y[x], x]
DSolve[{%, y[0.05] == 0.05, y[1.05] == 0.55}, y[x], x]
Show[finale0, Plot[y[x] /. %, {x, 0, 1}, PlotStyle → {Thick, Black}]]
```
pierwiastek kwadratowy; rozwiązywanie równań różniczkowych; pokaż; wykres; styl grafiki; gruby; czarny

Out[87]= $-\frac{y''(x)}{\left(y'(x)^2+1\right)^{3/2}}=0$

Out[88]= $\{\{y(x)\to 0.5\,x+0.025\}\}$

Out[89]=



Lagrangian i równania Eulera-Lagranga

Definiujemy działanie dla układu:

$S = \int_{t_A}^{t_B} L(u_1, u_2, \ldots, u'_1, u'_2)\, dt$

u_i -- współrzędena uogólniona
$u'_i = du_i/dt$ -- prędkość uogólniona

L = T (energia kinetyczna) - V(energia potencjalna)

Równania ruchu zadane są równianiami Eulera-Lagrange'a

$\frac{d}{dt}\frac{\partial}{\partial u'_i}L - \frac{\partial}{\partial u_i}L = 0$

Ale Wolfram Mathematica posiada pakiet, który dla zadanego Lagrangianiu i zmiennych uogólnionych sam wylicza równania...

Needs["VariationalMethods`"]

Podwójne wahadło matematyczne

In[90]:=
```
Clear[X1, Y1, x1, y1, X2, Y2, x2, y2]
wyczyść
X1 = l1 Sin[θ1[t]]
        sinus
Y1 = l1 Cos[θ1[t]]
        cosinus
x1[t_] := Normal[Series[X1, { θ1[t], 0, 2}]]
           norma⋯ szereg
y1[t_] := Normal[Series[Y1, { θ1[t], 0, 2}]]
           norma⋯ szereg
```

Out[91]= $l1\ \sin(\theta 1(t))$

Out[92]= $l1\ \cos(\theta 1(t))$

In[95]:=
```
X2 = x1[t] + l2 Sin[θ2[t]]
                sinus
Y2 = y1[t] + l2 Cos[θ2[t]]
                cosinus
x2[t_] := Normal[Series[X2, { θ2[t], 0, 2}]]
           norma⋯ szereg
y2[t_] := Normal[Series[Y2, { θ2[t], 0, 2}]]
           norma⋯ szereg
```

Out[95]= $l1\ \theta 1(t) + l2\ \sin(\theta 2(t))$

Out[96]= $-\frac{1}{2} l1\ \theta 1(t)^2 + l1 + l2\ \cos(\theta 2(t))$

In[99]:=
```
x1[t]
y1[t]
x2[t]
y2[t]
```

Out[99]= $l1\ \theta 1(t)$

Out[100]= $l1 - \frac{1}{2} l1\ \theta 1(t)^2$

Out[101]= $l1\ \theta 1(t) + l2\ \theta 2(t)$

Out[102]= $-\frac{1}{2} l1\ \theta 1(t)^2 + l1 - \frac{1}{2} l2\ \theta 2(t)^2 + l2$

In[103]:=
```
T = (m1 / 2) (D[x1[t], t]^2) + (m1 / 2) (D[y1[t], t]^2) +
              oblicz pochodną          oblicz pochodną
    (m2 / 2) (D[x2[t], t]^2) + (m2 / 2) (D[y2[t], t]^2) // Expand
              oblicz pochodną          oblicz pochodną     rozszerz
```

Out[103]=
$$\frac{1}{2} \mathrm{l1}^2\, \mathrm{m1}\, \theta1(t)^2\, \theta1'(t)^2 + \frac{1}{2} \mathrm{l1}^2\, \mathrm{m1}\, \theta1'(t)^2 + \frac{1}{2} \mathrm{l1}^2\, \mathrm{m2}\, \theta1(t)^2\, \theta1'(t)^2 + \frac{1}{2} \mathrm{l1}^2\, \mathrm{m2}\, \theta1'(t)^2 + \mathrm{l1}\, \mathrm{l2}\, \mathrm{m2}\, \theta1'(t)\, \theta2'(t) + \mathrm{l1}\, \mathrm{l2}\, \mathrm{m2}\, \theta1(t)\, \theta2(t)\, \theta1'(t)\, \theta2'(t) + \frac{1}{2} \mathrm{l2}^2\, \mathrm{m2}\, \theta2(t)^2\, \theta2'(t)^2 + \frac{1}{2} \mathrm{l2}^2\, \mathrm{m2}\, \theta2'(t)^2$$

In[104]:=
```
T /. {θ1[t]^2 θ1'[t]^2 → 0, θ2[t]^2 θ2'[t]^2 → 0, θ1[t] θ2[t] θ1'[t] → 0}
T = %
```

Out[104]=
$$\frac{1}{2} \mathrm{l1}^2\, \mathrm{m1}\, \theta1'(t)^2 + \frac{1}{2} \mathrm{l1}^2\, \mathrm{m2}\, \theta1'(t)^2 + \mathrm{l1}\, \mathrm{l2}\, \mathrm{m2}\, \theta1'(t)\, \theta2'(t) + \frac{1}{2} \mathrm{l2}^2\, \mathrm{m2}\, \theta2'(t)^2$$

Out[105]=
$$\frac{1}{2} \mathrm{l1}^2\, \mathrm{m1}\, \theta1'(t)^2 + \frac{1}{2} \mathrm{l1}^2\, \mathrm{m2}\, \theta1'(t)^2 + \mathrm{l1}\, \mathrm{l2}\, \mathrm{m2}\, \theta1'(t)\, \theta2'(t) + \frac{1}{2} \mathrm{l2}^2\, \mathrm{m2}\, \theta2'(t)^2$$

In[106]:=
```
V = -m1 g y1[t] - m2 g y2[t]
% /. {θ1[t] → 0, θ2[t] → 0}
V - % // Simplify
         uprość
V = %
```

Out[106]=
$$-9.81\, \mathrm{m2} \left(-\frac{1}{2} \mathrm{l1}\, \theta1(t)^2 + \mathrm{l1} - \frac{1}{2} \mathrm{l2}\, \theta2(t)^2 + \mathrm{l2}\right) - 9.81\, \mathrm{m1} \left(\mathrm{l1} - \frac{1}{2} \mathrm{l1}\, \theta1(t)^2\right)$$

Out[107]=
$$-9.81\, \mathrm{m2}\, (\mathrm{l1} + \mathrm{l2}) - 9.81\, \mathrm{l1}\, \mathrm{m1}$$

Out[108]=
$$\mathrm{l1}\, (4.905\, \mathrm{m1} + 4.905\, \mathrm{m2})\, \theta1(t)^2 + 4.905\, \mathrm{l2}\, \mathrm{m2}\, \theta2(t)^2 + 0.$$

Out[109]=
$$\mathrm{l1}\, (4.905\, \mathrm{m1} + 4.905\, \mathrm{m2})\, \theta1(t)^2 + 4.905\, \mathrm{l2}\, \mathrm{m2}\, \theta2(t)^2 + 0.$$

In[110]:=
```
sol3 = EulerEquations[T - V, {θ1[t], θ2[t]}, t]
```

Out[110]=
$$\{\mathrm{l1}\, (\mathrm{l1}\, (-1.\, \mathrm{m1} - 1.\, \mathrm{m2})\, \theta1''(t) - 1.\, \mathrm{l2}\, \mathrm{m2}\, \theta2''(t) + (-9.81\, \mathrm{m1} - 9.81\, \mathrm{m2})\, \theta1(t)) = 0, \; -1.\, \mathrm{l2}\, \mathrm{m2}\, (\mathrm{l1}\, \theta1''(t) + \mathrm{l2}\, \theta2''(t) + 9.81\, \theta2(t)) = 0\}$$

In[111]:=
```
sol3 = sol3 /. {m1 → m, m2 -> α m, l1 → l, l2 → β l}
```

Out[111]=
$$\{l\, (-1.\, \alpha\, \beta\, l\, m\, \theta2''(t) + l\, (-1.\, \alpha\, m - 1.\, m)\, \theta1''(t) + (-9.81\, \alpha\, m - 9.81\, m)\, \theta1(t)) = 0, \; -1.\, \alpha\, \beta\, l\, m\, (\beta\, l\, \theta2''(t) + l\, \theta1''(t) + 9.81\, \theta2(t)) = 0\}$$

In[112]:=

```
DSolve[
rozwiązywanie równań różniczkowych
  {sol3, θ1'[0] == 0, θ2'[0] == 0, θ1[0] == θ10, θ2[0] == θ20}, {θ1[t], θ2[t]}, t];
sol3F = Simplify[%, Assumptions → {g > 0, m > 0, l > 0, α > 0, β > 0}]
          uprość           założenia
```

Out[113]=

$$
\begin{aligned}
&\Big\{\Big\{\theta 1(t) \to \Big(\Big(\big((0.5\,\beta^4 + 2.\,\beta^3 + 3.\,\beta^2 + 2.\,\beta + 0.5)\,\alpha^5 + (2.5\,\beta^4 + 6.\,\beta^3 + 7.\,\beta^2 + 6.\,\beta + 2.5)\,\alpha^4 + \\
&\quad (5.\,\beta^4 + 4.\,\beta^3 + 6.\,\beta^2 + 4.\,\beta + 5.)\,\alpha^3 + (5.\,\beta^4 - 4.\,\beta^3 + 6.\,\beta^2 - 4.\,\beta + 5.)\,\alpha^2 + \\
&\quad (2.5\,\beta^4 - 6.\,\beta^3 + 7.\,\beta^2 - 6.\,\beta + 2.5)\,\alpha + 0.5\,\beta^4 - 2.\,\beta^3 + 3.\,\beta^2 - 2.\,\beta + 0.5\big) \\
&\quad \Big(l\,\alpha\,\theta 20\,\text{RootSum}\Big[0.0103911\,\#1^4 + 0.101937\,l\,\beta^2\,\#1^2 + 0.101937\,l\,\alpha\,\beta^2\,\#1^2 + \\
&\qquad 0.101937\,l\,\beta\,\#1^2 + 0.101937\,l\,\alpha\,\beta\,\#1^2 + 1.\,l^2\,\beta^3 + 1.\,l^2\,\alpha\,\beta^3\,\&, \\
&\qquad e^{\frac{1.\,t\,\#1}{l\,\beta}} \Big/ (1.\,l\,\beta^2 + 1.\,l\,\alpha\,\beta^2 + 1.\,l\,\beta + 1.\,l\,\alpha\,\beta + 0.203874\,\#1^2)\,\&\Big]\,\beta^2 + \\
&\quad \theta 10\,\text{RootSum}\Big[0.0103911\,\#1^4 + 0.101937\,l\,\beta^2\,\#1^2 + 0.101937\,l\,\alpha\,\beta^2\,\#1^2 + 0.101937\,l\,\beta\,\#1^2 + \\
&\qquad 0.101937\,l\,\alpha\,\beta\,\#1^2 + 1.\,l^2\,\beta^3 + 1.\,l^2\,\alpha\,\beta^3\,\&, \Big(0.101937\,e^{\frac{1.\,t\,\#1}{l\,\beta}}\,\#1^2 + 1.\,e^{\frac{1.\,t\,\#1}{l\,\beta}}\,l\,\beta + \\
&\qquad 1.\,e^{\frac{1.\,t\,\#1}{l\,\beta}}\,l\,\alpha\,\beta\Big) \Big/ (1.\,l\,\beta^2 + 1.\,l\,\alpha\,\beta^2 + 1.\,l\,\beta + 1.\,l\,\alpha\,\beta + 0.203874\,\#1^2)\,\&\Big]\Big)\Big) \Big/ \\
&\quad \big((1.\,\beta^4 + 4.\,\beta^3 + 6.\,\beta^2 + 4.\,\beta + 1.)\,\alpha^5 + (5.\,\beta^4 + 12.\,\beta^3 + 14.\,\beta^2 + 12.\,\beta + 5.)\,\alpha^4 + \\
&\quad (10.\,\beta^4 + 8.\,\beta^3 + 12.\,\beta^2 + 8.\,\beta + 10.)\,\alpha^3 + \\
&\quad (10.\,\beta^4 - 8.\,\beta^3 + 12.\,\beta^2 - 8.\,\beta + 10.)\,\alpha^2 + \\
&\quad (5.\,\beta^4 - 12.\,\beta^3 + 14.\,\beta^2 - 12.\,\beta + 5.)\,\alpha + 1.\,\beta^4 - 4.\,\beta^3 + 6.\,\beta^2 - 4.\,\beta + 1.\big), \\
&\theta 2(t) \to \Big(l\,\beta\,\big((0.5\,\beta^4 + 2.\,\beta^3 + 3.\,\beta^2 + 2.\,\beta + 0.5)\,\alpha^6 + (3.\,\beta^4 + 8.\,\beta^3 + 10.\,\beta^2 + 8.\,\beta + 3.)\,\alpha^5 + \\
&\quad (7.5\,\beta^4 + 10.\,\beta^3 + 13.\,\beta^2 + 10.\,\beta + 7.5)\,\alpha^4 + (10.\,\beta^4 + 12.\,\beta^2 + 10.)\,\alpha^3 + \\
&\quad (7.5\,\beta^4 - 10.\,\beta^3 + 13.\,\beta^2 - 10.\,\beta + 7.5)\,\alpha^2 + (3.\,\beta^4 - 8.\,\beta^3 + 10.\,\beta^2 - 8.\,\beta + 3.)\,\alpha + \\
&\quad 0.5\,\beta^4 - 2.\,\beta^3 + 3.\,\beta^2 - 2.\,\beta + 0.5\big)\,\theta 10\,\text{RootSum}\Big[0.0103911\,\#1^4 + 0.101937\,l\,\beta^2\,\#1^2 + \\
&\qquad 0.101937\,l\,\alpha\,\beta^2\,\#1^2 + 0.101937\,l\,\beta\,\#1^2 + 0.101937\,l\,\alpha\,\beta\,\#1^2 + 1.\,l^2\,\beta^3 + 1.\,l^2\,\alpha\,\beta^3\,\&, \\
&\qquad e^{\frac{1.\,t\,\#1}{l\,\beta}} \Big/ (1.\,l\,\beta^2 + 1.\,l\,\alpha\,\beta^2 + 1.\,l\,\beta + 1.\,l\,\alpha\,\beta + 0.203874\,\#1^2)\,\&\Big] + \\
&\quad \big((0.5\,\beta^4 + 2.\,\beta^3 + 3.\,\beta^2 + 2.\,\beta + 0.5)\,\alpha^5 + (2.5\,\beta^4 + 6.\,\beta^3 + 7.\,\beta^2 + 6.\,\beta + 2.5)\,\alpha^4 + \\
&\quad (5.\,\beta^4 + 4.\,\beta^3 + 6.\,\beta^2 + 4.\,\beta + 5.)\,\alpha^3 + (5.\,\beta^4 - 4.\,\beta^3 + 6.\,\beta^2 - 4.\,\beta + 5.)\,\alpha^2 + \\
&\quad (2.5\,\beta^4 - 6.\,\beta^3 + 7.\,\beta^2 - 6.\,\beta + 2.5)\,\alpha + 0.5\,\beta^4 - 2.\,\beta^3 + 3.\,\beta^2 - 2.\,\beta + 0.5\big) \\
&\quad \theta 20\,\text{RootSum}\Big[0.0103911\,\#1^4 + 0.101937\,l\,\beta^2\,\#1^2 + 0.101937\,l\,\alpha\,\beta^2\,\#1^2 + \\
&\qquad 0.101937\,l\,\beta\,\#1^2 + 0.101937\,l\,\alpha\,\beta\,\#1^2 + 1.\,l^2\,\beta^3 + 1.\,l^2\,\alpha\,\beta^3\,\&, \\
&\qquad \Big(1.\,e^{\frac{1.\,t\,\#1}{l\,\beta}}\,l\,\beta^2 + 1.\,e^{\frac{1.\,t\,\#1}{l\,\beta}}\,l\,\alpha\,\beta^2 + 0.101937\,e^{\frac{1.\,t\,\#1}{l\,\beta}}\,\#1^2\Big) \Big/ \\
&\qquad (1.\,l\,\beta^2 + 1.\,l\,\alpha\,\beta^2 + 1.\,l\,\beta + 1.\,l\,\alpha\,\beta + 0.203874\,\#1^2)\,\&\Big]\Big) \Big/ \\
&\quad \big((1.\,\beta^4 + 4.\,\beta^3 + 6.\,\beta^2 + 4.\,\beta + 1.)\,\alpha^5 + (5.\,\beta^4 + 12.\,\beta^3 + 14.\,\beta^2 + 12.\,\beta + 5.)\,\alpha^4 + \\
&\quad (10.\,\beta^4 + 8.\,\beta^3 + 12.\,\beta^2 + 8.\,\beta + 10.)\,\alpha^3 + \\
&\quad (10.\,\beta^4 - 8.\,\beta^3 + 12.\,\beta^2 - 8.\,\beta + 10.)\,\alpha^2 + \\
&\quad (5.\,\beta^4 - 12.\,\beta^3 + 14.\,\beta^2 - 12.\,\beta + 5.)\,\alpha + 1.\,\beta^4 - 4.\,\beta^3 + 6.\,\beta^2 - 4.\,\beta + 1.\big)\Big\}\Big\}
\end{aligned}
$$

In[114]:=
```
sol3F /. {g → 9.81, α → 1, β → 1, θ10 → #1, θ20 → #2, l → 1} & [0.03, 0.01];
Plot[{θ1[t] /. %, θ2[t] /. %}, {t, 0, 4 π}, AxesLabel → {t, θ}]
```
wykres oznaczenia osi

Out[115]=