# Radiative Corrections - Veltman-Passarino Integrals and Other Stuff

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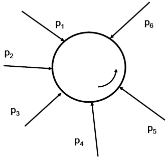
# Streszczenie

Celem seminarium jest wprowadzenie do rachunków radiacyjnych. Zacznę od omówienia podstawowych wielkości i tricków rachunkowych używanych w rachunkach radiacyjnych. Wreszcie opowiem o bieżących zastosowaniach, w szczególności o procesach w których wymieniane są dwa fotony a tarczą jest nukleon. Pokaże możliwości jakie dają pakiety FeynCalc i LoopTool. Pozwalają one prowadzić rachunki radiacyjne dla QED oraz modelu standardowego.

- Results of: G. 't Hooft anf M. Veltman, Nucl. Phys. **B**153 (1979) 365; G. Passarino, M. Veltman, Nucl. Phys. **B**160, 151 (1979).
- Electron Vertex Correction;
- FeynCalc, LoopTool and Numerical Evaluation of the Integrals;
- $ep \longrightarrow ep$ scattering and two-photon exchange.

- Accounting higher order contribution from the perturbation scheme;
- Infrared (QED) and Ultraviolet corrections, low and high limits of the effective theories;
- Loop diagrams, one loop, two , loops etc.;

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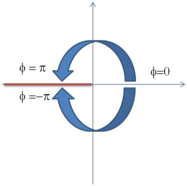
$$A(p_1^2, p_2^2, \ldots, p_6^2, p_1 \cdot p_2, \ldots, p_5 \cdot p_6) = \int \frac{d^n k}{(2\pi)^n} \frac{Nominator(\ldots)}{(k^2 - \mu_1^2 + i\epsilon)((p_1 + k)^2 - \mu_2^2 + i\epsilon)\ldots((p_1 + \ldots + p_5 + k)^2 - \mu_6^2 + i\epsilon)}$$

- R. E. Cutcovsky J. Math. Phys. **1**, 429; L. D. Landau, Nucl. Phys. **13**, 181; D.B. Melrose Nuovo. Cimento. 40, 181.
- Dispertion Analysis, see e.g. Pitajewski, Lifszyc, *Relatywistyczna Teoria Kwantów* vol II.
- Analytical and numerical evaluation;

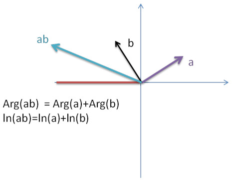
- Effective Field Theory, ultraviolet divergency **UV**, probably the theory does not properly describe the physics at short distance;
- QED, Infrared divergency **IR**, non locality of the theory;




- The logarithm has a cut along negative real axis.
- $$\begin{aligned} \ln(ab) &= \ln a + \ln b + \eta(a, b) & (1) \\ \eta(a, b) &= 2\pi i\,(\theta(-\mathrm{Im}a)\theta(-\mathrm{Im}b)\theta(\mathrm{Im}(ab)) - \theta(\mathrm{Im}a)\theta(\mathrm{Im}b)\theta(-\mathrm{Im}(ab))) & (2) \end{aligned}$$

Above rules become natural, when one draws the appropriate plots (see next pages).

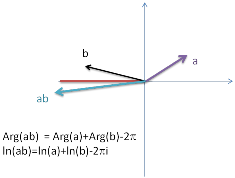
- A phase of a complex number ranges from $-\pi$ (below the cut line) to $\pi$ (above the cut line).
- If one goes from the top half of the complex plane into the bottom part of the complex plane, across the cut, then one has to add $-2\pi i$ to the phase argument.
- If the direction is opposite then needs to add $2\pi i$. Because limited range of the argument, it is impossible to cross the cut line more than once during single multiplication operation.




Suppose we have too complex numbers $a$ and $b$ with $\mathrm{Im}(a) > 0$, and $\mathrm{Im}(b) > 0$ then if $\mathrm{Im}(ab) > 0$, see Figure below, then the cut line is not crossed, hence

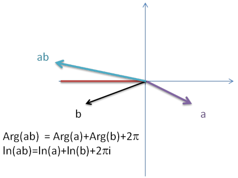
$$\mathrm{Arg}(ab) = \mathrm{Arg}(a) + \mathrm{Arg}(b), \quad \ln(ab) = \ln(a) + \ln(b). \tag{3}$$

Analogical property occurs for $\mathrm{Im}(a) < 0$, and $\mathrm{Im}(b) < 0$ then if $\mathrm{Im}(ab) < 0$.




Suppose we have too complex numbers $a$ and $b$ for which $\mathrm{Im}(a) > 0$, and $\mathrm{Im}(b) > 0$ but with $\mathrm{Im}(ab) < 0$, see Figure below, the cut line is crossed (from top to the bottom), hence,

$$\mathrm{Arg}(ab) = \mathrm{Arg}(a) + \mathrm{Arg}(b) - 2\pi, \quad \ln(ab) = \ln(a) + \ln(b) - 2\pi i. \tag{4}$$




Suppose we have to complex numbers $a$ and $b$ with $\mathrm{Im}(a) < 0$, and $\mathrm{Im}(b) < 0$, and with $\mathrm{Im}(ab) > 0$, see Fig. **??**- the cut line is crossed (from bottom to top), hence,

$$\mathrm{Arg}(ab) = \mathrm{Arg}(a) + \mathrm{Arg}(b) + 2\pi\,, \quad \ln(ab) = \ln(a) + \ln(b) + 2\pi i. \tag{5}$$

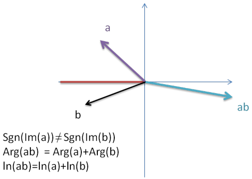



Figure :

Suppose we have to complex numbers $a$ and $b$ with $\mathrm{sgn}(\mathrm{Im}(a)) \neq \mathrm{sgn}(\mathrm{Im}(b))$, see Fig. 1, then it is obvious that multiplication $ab$ increases the total argument and the cut line of the logarithm is not crossed.

$$\mathrm{Arg}(ab) = \mathrm{Arg}(a) + \mathrm{Arg}(b), \quad \ln(ab) = \ln(a) + \ln(b). \tag{6}$$

Analogically it is the case, if $\mathrm{sgn}(\mathrm{Im}(a)) = \mathrm{sgn}(1/\mathrm{Im}(b))$, because $\mathrm{sgn}(\mathrm{Im}(a)) \neq \mathrm{sgn}(\mathrm{Im}(b))$.

Let $A$ and $B$ are real, and $\epsilon$ is infinitesimal, assume without loosing generality that $A \geq B$, then

$$\ln(AB - i\epsilon) = \ln(AB - i\epsilon(A - B)) = \ln((A + i\epsilon)(B - i\epsilon)) \tag{7}$$

because $A - B \geq 0$, hence

$$\ln(AB - i\epsilon) = \ln(A + i\epsilon) + \ln(B - i\epsilon) \tag{8}$$

Consider real $C > 0$

$$\ln(AB - iC) = \ln((A - i\epsilon)(B - iC/A)) \tag{9}$$

Consider the integral

$$I = \int_0^1 dx \frac{1}{ax + b}. \tag{10}$$

If $a$ and $b$ are real then it can be easy shown that

$$I = \frac{1}{a} \ln \left| 1 + \frac{a}{b} \right| . \tag{11}$$

Assume that $ax + b$ has definite sign of the imaginary part in the all $x$ range then

$$I = \int_0^1 dx \frac{1}{ax + b} = \frac{1}{a} \int_0^1 dx \frac{1}{x + \frac{b}{a}} = \frac{1}{a} \ln \left( x + \frac{b}{a} \right) \Big|_0^1 = \frac{1}{a} \left\{ \ln \left( 1 + \frac{b}{a} \right) - \ln \left( \frac{b}{a} \right) \right\} \tag{12}$$

because the imaginary parts of $1 + b/a$ and $b/a$ have the same sign, then

$$I = \ln(1 + a/b). \tag{13}$$

**It is the same result as for the conventional case.**

Di-logarithm, or Spence function is defined as its follows:

$$Sp(x) = -\int_0^1 dt\,\frac{\ln(1-xt)}{t} = -\int_0^x dt\,\frac{\ln(1-t)}{t} = \underbrace{\int_0^{-\ln(1-x)} \frac{u}{e^u - 1}\,du}_{\to Bernouilly} = L_2(x) \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^2} \tag{14}$$

$$0 = -Sp(x) - Sp(1-x) + \frac{1}{6}\pi^2 - \ln(x)\ln(1-x) \tag{15}$$

$$Sp(x) = \sum_{n=0}^{\infty} B_n\,\frac{z^{n+1}}{(n+1)!}, \quad \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n\,\frac{x^n}{n!} \tag{16}$$

$B_n$ Bernouilly numbers

$$\lim_{\mathrm{Re}\lambda\to 0} \mathrm{Li}_2(-e^{\lambda}) = -\frac{1}{2}\lambda^2, \quad \Rightarrow \lim_{x\to\infty} \mathrm{Li}_2(-1/x) = -\frac{1}{2}\ln^2(1/x) \tag{17}$$

Nice review: L. C. Maximon, Proc. R. Soc. Lond. **A459**, 2807 (2003).

$$\frac{1}{A_1 \dots A_n} = \left[\prod_{k=1}^{n} \int_0^1 dx_k\right] \frac{(n-1)!}{\left[x_1 A_1 + \dots + x_n A_n\right]^n} \delta(1 - x_1 - \dots - x_n) \tag{18}$$

$$= \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \dots \int_0^{1-x_1-\dots-x_{n-2}} dx_{n-1} \frac{(n-1)!}{\left[x_1 A_1 + \dots + x_n A_n\right]^n} \tag{19}$$

$$\frac{1}{A_1^{\alpha_1} \dots A_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 + \dots \alpha_n)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} \prod_{k=1}^{n} \left[\int_0^1 dx_k\right] \frac{x_1^{\alpha_1 - 1} \dots x_n^{\alpha_n - 1}}{\left[x_1 A_1 + \dots + x_n A_n\right]^{\alpha_1 + \dots + \alpha_n}} \delta(1 - x_1 - \dots - x_n) \tag{20}$$

$$\frac{1}{A_1 \dots A_n} = \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-2}} dx_{n-1} \frac{(n-1)!}{\left[x_{n-1} A_1 + (x_{n-2} - x_{n-1}) A_2 \dots + (1 - x_1) A_n\right]^n} \tag{21}$$

$$\frac{1}{A_1 \dots A_n} = \int_0^1 x_1^{n-2} dx_1 \int_0^1 x^{n-3} dx_2 \dots \int_0^1 dx_{n-1} \frac{(n-1)!}{\left[x_1 \cdot \dots \cdot x_{n-1} A_1 + x_1 \cdot \dots \cdot x_{n-2}(1 - x_{n-1}) A_2 \dots + (1 - x_1) A_n\right]^n} \tag{22}$$

$$\frac{1}{(a+\lambda)(b+\lambda)} = -\frac{\partial}{\partial \lambda} \int_0^1 d\beta \frac{1}{[\beta a + (1-\beta) b + \lambda]} \tag{23}$$

## Standard Formulae: $\mathcal{I}_0$

$$\mathcal{I}_0(n,k) = \int \frac{d^n l}{(2\pi)^n} \frac{1}{(l^2 - \Delta + i\epsilon)^k} \tag{24}$$

after Wick rotation, $k_0 = ik_4$,

$$\mathcal{I}_0(n,k) = (-1)^k i \int \frac{d^n l}{(2\pi)^n} \frac{1}{(l^2 + \Delta - i\epsilon)^k} \tag{25}$$

$$= (-1)^k i \frac{1}{(2\pi)^n} \int d\Omega_n \int_0^\infty dl \frac{l^{n-1}}{(l^2 + \Delta - i\epsilon)^k} \tag{26}$$

$$= (-1)^k i \frac{1}{(2\pi)^n} \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty dl \frac{l^{n-1}}{(l^2 + \Delta - i\epsilon)^k} \tag{27}$$

$$= (-1)^k i \frac{1}{(2\pi)^n} \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty dl^2 \frac{l^{\frac{n-2}{2}}}{(l^2 + \Delta - i\epsilon)^k} \tag{28}$$

where

$$\int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{29}$$

$$\Gamma(z) = \int_0^1 dt t^{z-1} e^{-t} \tag{30}$$

Gamma function around $x = 0$ behaves as

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x) \tag{31}$$

around $x = -n$, $n = 1, 2, \ldots$

$$\Gamma(x) = \frac{(-1)^n}{n!} \left( \frac{1}{x+n} - \gamma + 1 + \ldots + \frac{1}{n} + \mathcal{O}(x+n) \right) \tag{32}$$

Let's introduce a new integration variable

$$\chi = \frac{\Delta - i\epsilon}{l^2 + \Delta - i\epsilon}, \quad l^2 + \Delta - i\epsilon = \frac{\Delta - i\epsilon}{\chi}, \quad l^2 = (\Delta - i\epsilon)\frac{1-\chi}{\chi}, \quad dl^2 = -\frac{\Delta - i\epsilon}{\chi^2} \tag{33}$$

then

$$\mathcal{I}_0(n,k) = \frac{i(-1)^k}{(2\pi)^n}\frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^1 d\chi \frac{\Delta - i\epsilon}{\chi^2}(\Delta - i\epsilon)^{\frac{n}{2}-1}\left(\frac{1-\chi}{\chi}\right)^{\frac{n}{2}-1}\frac{\chi^k}{(\Delta - i\epsilon)^k} \tag{34}$$

$$= \frac{i(-1)^k}{(2\pi)^n}\frac{\pi^{n/2}}{\Gamma(n/2)}(\Delta - i\epsilon)^{\frac{n}{2}-k}\int_0^1 d\chi\,(1-\chi)^{\frac{n}{2}-1}\chi^{-\frac{n}{2}+k-1} \tag{35}$$

$$= \frac{i(-1)^k}{(2\pi)^n}\frac{\pi^{n/2}}{\Gamma(n/2)}(\Delta - i\epsilon)^{\frac{n}{2}-k} Beta\left(\frac{n}{2}, k - \frac{n}{2}\right) \tag{36}$$

$$= \frac{i(-1)^k}{(2\pi)^n}\frac{\pi^{n/2}}{\Gamma(n/2)}(\Delta - i\epsilon)^{\frac{n}{2}-k}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(k-\frac{n}{2}\right)}{\Gamma(k)} \tag{37}$$

$$= (-1)^k \frac{i}{2^n \pi^{\frac{n}{2}}}\frac{\Gamma\left(k-\frac{n}{2}\right)}{(k-1)!}(\Delta - i\epsilon)^{\frac{n}{2}-k} \tag{38}$$

$$Beta(\alpha,\beta) = \int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{39}$$

$$\mathcal{I}_2(n, k) = \int \frac{d^n l}{(2\pi)^n} \frac{l^2}{(l^2 - \Delta + i\epsilon)^k} = \mathcal{I}_0(n, k-1) + (\Delta - i\epsilon)\mathcal{I}_0(n, k) \quad (40)$$

$$= (-1)^{k-1} \frac{in}{2^{n+1}\pi^{\frac{n}{2}}} \frac{\Gamma\left(k - \frac{n}{2} - 1\right)}{\Gamma(k)} (\Delta - i\epsilon)^{\frac{n}{2}-k+1} \quad (41)$$

It is easy to see that,

$$\int \frac{d^n l'}{(2\pi)^n} \frac{l'^{\mu}}{[l'^2 - \Delta + i\epsilon]^3} = 0 \tag{42}$$

$$\int \frac{d^n l'}{(2\pi)^n} \frac{l'^{\mu} l^{\nu}}{[l'^2 - \Delta + i\epsilon]^3} = \frac{g^{\mu\nu}}{n} \int \frac{d^n l'}{(2\pi)^n} \frac{l'^2}{[l'^2 - \Delta + i\epsilon]^3}. \tag{43}$$

It is obvious that the first integral vanishes, from the symmetry property. The same symmetry property constrains the result of the second integral i.e. it vanishes for $\nu \neq \mu$. It is obvious from the symmetry property that

$$\int \frac{d^n l'}{(2\pi)^n} \frac{l'^{i} l^{i}}{[l'^2 - \Delta + i\epsilon]^3} = \int \frac{d^n l'}{(2\pi)^n} \frac{l'^{j} l^{j}}{[l'^2 - \Delta + i\epsilon]^3} = \frac{1}{3} \int \frac{d^n l'}{(2\pi)^n} \frac{\mathbf{l}^2}{[l'^2 - \Delta + i\epsilon]^3} \tag{44}$$

hence it is obvious from the symmetry property that

$$\int \frac{d^n l'}{(2\pi)^n} \frac{l'^{\mu} l^{\nu}}{[l'^2 - \Delta + i\epsilon]^3} = A g^{\mu\nu} \tag{45}$$

Contracting this result with the metric we get

$$An = \int \frac{d^n l'}{(2\pi)^n} \frac{l'^2}{[l'^2 - \Delta + i\epsilon]^3} \tag{46}$$

$$T^N_{\mu_1 \ldots \mu_P}(p_1, \ldots, p_{N-1}, m_0, \ldots, m_{N-1}) = \int d^D q \frac{q_{\mu_1} \cdots q_{\mu_P}}{D_0 D_1 \ldots D_{N-1}} \tag{47}$$

where

$$D_0 = q^2 - m_0^2 + i\epsilon. \quad D_i = (q + p_i)^2 - m_i^2 + i\epsilon, \quad i = 1, 2, \ldots, N-1. \tag{48}$$

$p_{i0} = p_i$ Notation, $T^N$ denotes the $N$th character of the alphabet, A, B, C, D

$$A \equiv T^1 \tag{49}$$

$$B \equiv T^2 \tag{50}$$

$$C \equiv T^3 \tag{51}$$

$$D \equiv T^4 \tag{52}$$

see: A. Denner, Fortschr.Phys.41:307-420,1993

Lorentz covariance of the integrals

$$q_{\mu_1} q_{\mu_2} \quad \to \quad g_{\mu_1 \mu_2} \tag{53}$$

$$q_{\mu_1} q_{\mu_2} q_{\mu_3} q_{\mu_4} \quad \to \quad g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} + g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} + g_{\mu_1 \mu_4} g_{\mu_2 \mu_3} \tag{54}$$

Let's introduce the notation

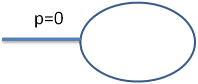
$$p_{i0} = p_i \,, \quad p_{ij} - p_i - p_j \tag{55}$$

$$
\begin{aligned}
B_\mu &= p_{1\mu} B_1, \\
B_{\mu\nu} &= g_{\mu\nu} B_{00} + p_{1\mu} p_{1\nu} B_{11}, \\
C_\mu &= p_{1\mu} C_1 + p_{2\mu} C_2 = \sum_{i=1}^{2} p_{i\mu} C_i, \\
C_{\mu\nu} &= g_{\mu\nu} C_{00} + p_{1\mu} p_{1\nu} C_{11} + p_{2\mu} p_{2\nu} C_{22} + (p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}) C_{12} \\
&= g_{\mu\nu} C_{00} + \sum_{i,j=1}^{2} p_{i\mu} p_{j\nu} C_{ij}, \\
C_{\mu\nu\rho} &= (g_{\mu\nu} p_{1\rho} + g_{\nu\rho} p_{1\mu} + g_{\mu\rho} p_{1\nu}) C_{001} + (g_{\mu\nu} p_{2\rho} + g_{\nu\rho} p_{2\mu} + g_{\mu\rho} p_{2\nu}) C_{002} \\
&\quad + p_{1\mu} p_{1\nu} p_{1\rho} C_{111} + p_{2\mu} p_{2\nu} p_{2\rho} C_{222} \\
&\quad + (p_{1\mu} p_{1\nu} p_{2\rho} + p_{1\mu} p_{2\nu} p_{1\rho} + p_{2\mu} p_{1\nu} p_{1\rho}) C_{112} \\
&\quad + (p_{2\mu} p_{2\nu} p_{1\rho} + p_{2\mu} p_{1\nu} p_{2\rho} + p_{1\mu} p_{2\nu} p_{2\rho}) C_{122} \\
&= \sum_{i=1}^{2} (g_{\mu\nu} p_{i\rho} + g_{\nu\rho} p_{i\mu} + g_{\mu\rho} p_{i\nu}) C_{00i} + \sum_{i,j,k=1}^{2} p_{i\mu} p_{j\nu} p_{k\rho} C_{ijk},
\end{aligned}
$$

taken from Fortschr.Phys.41:307-420,1993.

$$
\begin{aligned}
D_{\mu} &= \sum_{i=1}^{3} p_{i\mu} D_i, \\
D_{\mu\nu} &= g_{\mu\nu} D_{00} + \sum_{i,j=1}^{3} p_{i\mu} p_{j\nu} D_{ij}, \\
D_{\mu\nu\rho} &= \sum_{i=1}^{3} (g_{\mu\nu} p_{i\rho} + g_{\nu\rho} p_{i\mu} + g_{\mu\rho} p_{i\nu}) D_{00i} + \sum_{i,j,k=1}^{3} p_{i\mu} p_{j\nu} p_{k\rho} D_{ijk}, \\
D_{\mu\nu\rho\sigma} &= (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) D_{0000} \\
&\quad + \sum_{i,j=1}^{3} (g_{\mu\nu} p_{i\rho} p_{j\sigma} + g_{\nu\rho} p_{i\mu} p_{j\sigma} + g_{\mu\rho} p_{i\nu} p_{j\sigma} \\
&\qquad + g_{\mu\sigma} p_{i\nu} p_{j\rho} + g_{\nu\sigma} p_{i\mu} p_{j\rho} + g_{\rho\sigma} p_{i\mu} p_{j\nu}) D_{00ij} \\
&\quad + \sum_{i,j,k,l=1}^{3} p_{i\mu} p_{j\nu} p_{k\rho} p_{l\sigma} D_{ijkl}.
\end{aligned}
$$

taken from Fortschr.Phys.41:307-420,1993.




$$A(m^2) = \int d^n q \frac{1}{q^2 + m^2 - i\epsilon} = \mathcal{I}(n, 1), \tag{56}$$

$$A(m^2) = i\frac{\pi^{n/2}}{\Gamma(n/2)}(m^2 - i\epsilon)^{\frac{n}{2}-1}\frac{\Gamma\left(1-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(1\right)} = i\pi^{n/2}(m^2 - i\epsilon)^{\frac{n}{2}-1}\Gamma\left(1-\frac{n}{2}\right) \tag{57}$$

Now we do the dimensional regularization of the above integral, we take $n = 4 - \delta$

$$\pi^{\frac{n}{2}} = \pi^{2-\frac{\delta}{2}} = \exp\left[\left(2-\frac{\delta}{2}\right)\ln\pi\right] = \pi^2\left[1-\frac{\delta}{2}\ln\pi\right] \tag{58}$$

analogically

$$(m^2 - i\epsilon)^{\frac{n}{2}-1} = (m^2 - i\epsilon)^{1-\frac{\delta}{2}} = (m^2 - i\epsilon)\left[1-\frac{\delta}{2}\ln(m^2 - i\epsilon)\right] \tag{59}$$

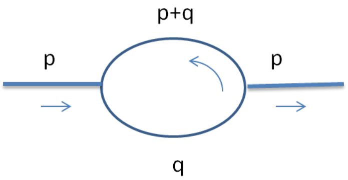
$$\Gamma(1-\frac{n}{2}) = \Gamma(-1+\frac{\delta}{2}) = -\left(\frac{2}{\delta} - \gamma + 1 + \mathcal{O}(\delta)\right) \tag{60}$$

Therefore

$$A(m^2) = -i\pi^2\left[1-\frac{\delta}{2}\ln\pi\right](m^2 - i\epsilon)\left[1-\frac{\delta}{2}\ln(m^2 - i\epsilon)\right]\left(\frac{2}{\delta} - \gamma + 1 + \mathcal{O}(\delta)\right) \tag{61}$$

$$= -\frac{2i\pi^2 m^2}{\delta} - i\pi^2 m^2 + \underbrace{i\pi^2 m^2(\gamma + \ln\pi)}_{*} + i\pi^2 m^2\ln(m^2 - i\epsilon) \tag{62}$$

In $*$ I have different expression that 't Hooft and Veltman they have $i\pi^2 m^2(-\gamma + \ln\pi)$!




$$B(k, m_1, m_2) = \int d^n q \frac{1}{(q^2 + m_1^2 - i\epsilon)((q+k)^2 + m_2^2 - i\epsilon)} \tag{63}$$

$$\begin{aligned} B(k, m_1, m_2) &= \int d^n q \frac{1}{(q^2 + m_1^2 - i\epsilon)((q+k)^2 + m_2^2 - i\epsilon)} && (64) \\ &= \int_0^1 dx \int d^n q \frac{1}{[q^2 + 2xk \cdot q + (1-x)m_1^2 + xk^2 + xm_2^2 - i\epsilon]^2} && (65) \\ &= \int_0^1 dx \int d^n q \frac{1}{[q^2 + (1-x)m_1^2 + x(1-x)k^2 + xm_2^2 - i\epsilon]^2} && (66) \end{aligned}$$

Similarly as in the previous section we make the change of the variables $k_0 \rightarrow ik_4$, so we have Euclidian coordinate system.

$$\begin{aligned} B(k, m_1, m_2) &= i \int_0^1 dx \int \frac{d^n q_E}{[q_E^2 + \Delta]^2} = \int_0^1 dx \mathcal{I}_0(n, 2)(\Delta) && (67) \\ \Delta &= (1-x)m_1^2 + x(1-x)k^2 + xm_2^2 - i\epsilon && (68) \end{aligned}$$

$$B(k, m_1, m_2) \quad = \quad i\pi^{n/2}\Gamma(-n/2+2)\int_0^1 dx\Delta^{n/2-2} \tag{69}$$

Now we do the dimensional regularization as in the previous section, in the case the one-point function:

$$B(k, m_1, m_2) \quad = \quad i\pi^2\left[1-\frac{\delta}{2}\ln\pi\right]\left(\frac{2}{\delta}-\gamma+\mathcal{O}(\delta)\right)\int_0^1 dx\left[1-\frac{\delta}{2}\ln\Delta\right] \tag{70}$$

$$= \quad \frac{2i\pi^2}{\delta} - \underbrace{i\pi^2(\gamma+\ln\pi)}_{*} - i\pi^2\int_0^1 dx\ln\left(\underbrace{(1-x)m_1^2+x(1-x)k^2+xm_2^2-i\epsilon}_{**}\right) \tag{71}$$

where

$$\Gamma(-n/2+2) = \Gamma\left(\frac{\delta}{2}\right) = \left(\frac{2}{\delta}-\gamma+\mathcal{O}(\delta)\right) \tag{72}$$

and

$$\Delta^{n/2-2} = 1-\frac{\delta}{2}\ln\Delta \tag{73}$$

Similarly as in the case of the one-point function I have some sign difference in $*$.

Let $x_1$ and $x_2$ are the roots of $**$ then

$$** = -k^2(x - x_1)(x - x_2), \quad x_{1,2} = \frac{-k^2 - m_2^2 + m_1^2 \pm \sqrt{\Delta_x}}{-2k^2} \tag{74}$$

- If $\Delta_x < 0$, then $\text{Im}(x_1)$ and $\text{Im}(x_2)$ have different signs.
- If $\Delta_x > 0$ ** can be written as

$$** = -k^2(x - x_1 + i\epsilon)(x - x_2 - i\epsilon) \tag{75}$$

and $\text{Im}(x_1 \to x_1 - i\epsilon)$ and $\text{Im}(x_2 \to x_2 + i\epsilon)$ have also different signs.

$$\int_0^1 dx \ln\left(-k^2(x - x_1)(x - x_2)\right) = \ln\left(-k^2\right) + \sum_{i=1}^{2} \int_0^1 dx \ln(x - x_i) \tag{76}$$

$$= \ln\left(-k^2\right) + \sum_{i=1}^{2} \left[(x - x_1) \ln(x - x_i) - x\right]_0^1 \tag{77}$$

$$= \ln\left(-k^2\right) - 2 + \ln(1 - x_1) + \ln(1 - x_2) + x_1 \ln\left(\frac{x_1 - 1}{x_1}\right) + x_2 \ln\left(\frac{x_2 - 1}{x_2}\right) \tag{78}$$

In above evaluation we forget about $i\epsilon$, notice that for $\Delta_x < 0$, $** > 0$, hance

$$** = (-k^2 - i\epsilon)(x - x_1)(x - x_2) \tag{79}$$

It does not have to be the case for $\Delta_x > 0$, than it may happen that $** < 0$. In this case its safer to write

$$\int_0^1 dx \ln\left(-k^2(x - x_1 + i\epsilon)(x - x_2 - i\epsilon)\right) = \ln\left(-k^2\right) + \int_0^1 dx \ln\left((x - x_1 + i\epsilon)(x - x_2 - i\epsilon)\right) \tag{80}$$

$$= \ln\left(-k^2\right) - 2 + (1 - x_1)\ln(1 - x_1 + i\epsilon) + (1 - x_2)\ln(1 - x_2 - i\epsilon) + x_1 \ln(i\epsilon - x_1) + x_2 \ln(-x_2 - i\epsilon) \tag{81}$$

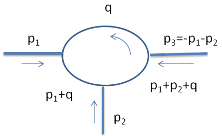
$$= \ln\left(-k^2\right) - 2 + \ln(1 - x_1 + i\epsilon) + \ln(1 - x_2 - i\epsilon) + x_1 \ln\left(\frac{x_1 - 1 - i\epsilon}{x_1 - i\epsilon}\right) + x_2 \ln\left(\frac{x_2 + i\epsilon - 1}{x_2 + i\epsilon}\right) \tag{82}$$

- If $k^2 = 0$ then there is only one root and $\ln(-k^2 - i\epsilon)$ must be replaced by $\ln(m_2^2 - m_1^2 - i\epsilon)$;
- If in addition $m_1 = m_2$ then there are no roots, and the integral reads

$$\ln(m_1^2 - i\epsilon)$$

- If in addition also $m_1 = 0$ we have infrared divergency, which should be treated in the proper way.

$$B(k, m_1, m_2) = \frac{2i\pi^2}{\delta} - \underbrace{i\pi^2(\gamma + \ln\pi)}_{*} - i\pi^2 \Bigg[\ln\left(-k^2\right) - 2 + \ln(1 - x_1 + i\epsilon) + \ln(1 - x_2 - i\epsilon) + x_1 \ln\left(\frac{x_1 - 1 - i\epsilon}{x_1 - i\epsilon}\right) + x_2 \ln\left(\frac{x_2 + i\epsilon - 1}{x_2 + i\epsilon}\right)\Bigg] \tag{83}$$




$$C(p_1, p_2, m_1, m_2, m_3) = \int d^n q \frac{1}{(q^2 + m_1^2 - i\epsilon)((q + p_1) + m_2^2 - i\epsilon)((q + p_1 + p_2) + m_3^2 - i\epsilon)} \tag{84}$$

The integral is a function of 6 variables $p_1^2$, $p_2^2$, $(p_1 + p_2)^2$, $m_1^2$, $m_2^2$ and , $m_3^2$
The integral can be written as

$$C(p_1, p_2, m_1, m_2, m_3) = \int_0^1 dx \int_0^x dy \int d^n q \frac{1}{D^3}, \tag{85}$$

where

$$D = yD_1 + (x - y)D_2 + (1 - x)D_3 \tag{86}$$

$$D_1 = q^2 + m_1^2 - i\epsilon \tag{87}$$

$$D_2 = (q + p_1)^2 + m_2^2 - i\epsilon \tag{88}$$

$$D_3 = (q + p_1 + p_2)^2 + m_3^2 - i\epsilon \tag{89}$$

$$D = yD_1 + (x-y)D_2 + (1-x)D_3 \quad (90)$$

$$= q^2 + 2q((1-y)p_1 + p_2(1-x)) + m_1^2 y + m_2^2 x - m_2^2 y - m_3^2 x + m_3^2 - p_1^2 y + p_1^2 - 2p_1 p_2 x + 2p_1 p_2 - p_2^2 x + p_2^2 \quad (91)$$

after change of the variables $q \longrightarrow q + (1-y)p_1 + (1-x)p_2$, we have

$$D = q^2 + ax^2 + by^2 + cxy + dx + ey + f \quad (92)$$

where

$$a = -p_2^2 \quad (93)$$

$$b = -p_1^2 \quad (94)$$

$$c = -2p_1 \cdot p_2 = -(p_1+p_2)^2 + p_1^2 + p_2^2 \quad (95)$$

$$d = (m_2^2 - m_3^2 + p_2^2) \quad (96)$$

$$e = (m_1^2 - m_2^2 + p_1^2 + 2p_1 p_2) \quad (97)$$

$$f = m_3^2 - i\epsilon \quad (98)$$

Let

$$\Delta = ax^2 + by^2 + cxy + dx + ey + f$$

then

$$C(p_1, p_2, m_1, m_2, m_3) = i \int_0^1 dx \int_0^x dy \int d\Omega_n \int_0^\infty dq_E \, \frac{q_E^{n-1}}{(q_E^2 + \Delta)^3} \quad (99)$$

$$= i \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^1 dx \int_0^x dy \int_0^\infty d(q_E^2) \frac{(q_E)^{n/2-1}}{(q_E^2 + \Delta)^3} \quad (100)$$

$$\int_0^\infty d(q_E^2)\frac{(q_E)^{n/2-1}}{(q_E^2+\Delta)^3} = \int_0^1 dz\frac{\Delta}{z^2}\left(\frac{(1-z)}{z}\Delta\right)^{n/2-1}\left(\frac{z}{\Delta}\right)^3 \quad (101)$$

$$= \Delta^{n/2-3}\frac{\Gamma(3-n/2)\Gamma(n/2)}{2} \quad (102)$$

$$(103)$$

Notice that

$$\Gamma(3-n/2) = \Gamma(1+\delta/2) = 1$$

Hence around $n = 4$ we have

$$\int_0^\infty d(q_E^2)\frac{(q_E)^{n/2-1}}{(q_E^2+\Delta)^3} = \Delta^{-1}\frac{\Gamma(n/2)}{2} \quad (104)$$

$$(105)$$

Eventually we get

$$C(p_1,p_2,m_1,m_2,m_3) = i\frac{\pi^2}{2}\int_0^1 dx\int_0^x dy\,\frac{1}{ax^2+by^2+cxy+dx+ey+f} = i\frac{\pi^2}{2}\int_0^1 dx\int_0^x dy\,\frac{1}{D'} \quad (106)$$

the result of 't Hooft and Veltman is twice larger?

The problem of above integral is the appearance of the two quadratic contributions $x^2$ and $y^2$, one can evaluate one of them but we still stay with another one, and further calculations become difficult. There is one standard trick, we make the change of the variables, namely

$$y = y' + \alpha x \tag{107}$$

the denominator of the integrals reads:

$$D' = (b\alpha^2 + c\alpha + a)x^2 + by'^2 + xy'(c + 2\alpha b) + x(d + e\alpha) + ey' + f \tag{108}$$

where $\alpha$ parameter is one of the roots of the equation

$$b\alpha^2 + c\alpha + a = 0. \tag{109}$$

then the denominator reads

$$D' = by'^2 + xy'(c + 2\alpha b) + x(d + e\alpha) + ey' + f \tag{110}$$

Now we are going to perform the series of the change of the variables

$$\int_0^1 dx \int_0^x dy = \int_0^1 dx \int_{-\alpha x}^{(1-\alpha)x} dy' = \int_{-\alpha}^0 dy' \int_{-\frac{y'}{\alpha}}^1 dx + \int_0^{1-\alpha} dy' \int_{\frac{y'}{1-\alpha}}^1 dx \tag{111}$$

Notice that alternatively, we would make different change of the variables, namely

$$x' = \alpha y + x \tag{112}$$

then

$$D' = ax'^2 + (b + c\alpha + a\alpha^2)y + (c + 2\alpha a)yx' + dx' + (\alpha d + e)y + f \tag{113}$$

(114)

$$C(p_1, p_2, m_1, m_2, m_3) = i\frac{\pi^2}{2}\left[\int_{-\alpha}^{0} dy' \frac{1}{y'(c+2\alpha b)+d+e\alpha} \ln(by'^2 + xy'(c+2\alpha b) + x(d+e\alpha) + ey' + f)\Big|_{-\frac{y'}{\alpha}}^{1}\right.$$
$$\left. + \int_0^{1-\alpha} dy' \frac{1}{y'(c+2\alpha b)+d+e\alpha} \ln(by'^2 + xy'(c+2\alpha b) + x(d+e\alpha) + ey' + f)\Big|_{\frac{y'}{1-\alpha}}^{1}\right]$$
$$= i\frac{\pi^2}{2}\left[\int_{-\alpha}^{0} dy' \frac{1}{y'(c+2\alpha b)+d+e\alpha} \ln \frac{by'^2 + y'(c+2\alpha b+e) + d + e\alpha + f}{by'^2 - y'^2(c+2\alpha b)/\alpha - y'(d+e\alpha)/\alpha + ey' + f}\right. \quad (115)$$
$$\left. + \int_0^{1-\alpha} dy' \frac{1}{y'(c+2\alpha b)+d+e\alpha} \ln \frac{by'^2 + y'(c+2\alpha b) + d + e\alpha + ey' + f}{by'^2 + y'^2(c+2\alpha b)/(1-\alpha) + y'(d+e\alpha)/(1-\alpha) + ey' + f}\right]$$
$$(116)$$

In the red integral we make the change of the variables $z = -y'/\alpha$, while in the second integral we introduce the variables $z = y'/(1-\alpha)$.

$$C(p_1, p_2, m_1, m_2, m_3) = i\frac{\pi^2}{2}\left[\int_0^1 dz \frac{\alpha}{-\alpha z(c+2\alpha b)+d+e\alpha} \ln \frac{b\alpha^2 z^2 - z\alpha[c+2\alpha b+e] + d + e\alpha + f}{z^2[b\alpha - (c+2\alpha b)]\alpha + z(d+e\alpha-\alpha e) + f}\right. \quad (117)$$
$$\left. + \int_0^1 dz \frac{1-\alpha}{(1-\alpha)(c+2\alpha b)+d+e\alpha} \ln \frac{b(1-\alpha)^2 z^2 + z(1-\alpha)(c+2\alpha b+e) + d + e\alpha + f}{z^2(1-\alpha)[b(1-\alpha) + (c+2\alpha b)] + z(d+e\alpha+e(1-\alpha)) + f}\right]$$
$$(118)$$

Now we are coming back to (115)

$$
\begin{aligned}
C(p_1, p_2, m_1, m_2, m_3) \quad = \quad & i\frac{\pi^2}{2}\left[\int_{-\alpha}^{1-\alpha} dy' \frac{1}{N(y')} \ln(by'^2 + ey' + f + N(y')) \right. && (119)\\
& - \int_{-\alpha}^{0} dy' \frac{1}{N(y')} \ln\left(by'^2 + ey' + f - \frac{y'}{\alpha}N(y')\right) && (120)\\
& \left. - \int_{0}^{1-\alpha} dy' \frac{1}{N(y')} \ln\left(by'^2 + ey' + f + \frac{y'}{1-\alpha}N(y')\right)\right] && (121)\\
& && (122)
\end{aligned}
$$

where

$$N(y') = y'(c + 2\alpha b) + d + \alpha e \tag{123}$$

Notice that expression $1/N(y')$ has singularity at

$$y_0 = -\frac{d + \alpha e}{c + 2\alpha b} \tag{124}$$

In order to have residuum equal to zero at $y_0$ we add to every integral the expression

$$-\ln(by_0^2 + ey_0 + f) \tag{125}$$

It can be done in the way that totally we add zero! Namely

$$C(p_1, p_2, m_1, m_2, m_3) = i\frac{\pi^2}{2}\left[\int_{-\alpha}^{1-\alpha} dy' \frac{1}{N(y')}\left\{\ln(by'^2 + ey' + f + N(y')) - \ln(by_0^2 + ey_0 + f)\right\}\right. \tag{126}$$

$$-\int_{-\alpha}^{0} dy' \frac{1}{N(y')}\left\{\ln\left(by'^2 + ey' + f - \frac{y'}{\alpha}N(y')\right) - \ln(by_0^2 + ey_0 + f)\right\} \tag{127}$$

$$\left.-\int_{0}^{1-\alpha} dy' \frac{1}{N(y')}\left\{\ln\left(by'^2 + ey' + f + \frac{y'}{1-\alpha}N(y')\right) - \ln(by_0^2 + ey_0 + f)\right\}\right] \tag{128}$$

This additional term allows studying the integrals with complex $\alpha$! Now we make the substitution $y' = y - \alpha$, $y = -y'/\alpha$ and $y = y'/(1-\alpha)$.

$$C(p_1, p_2, m_1, m_2, m_3) = i\frac{\pi^2}{2}\left[\int_0^1 \frac{dy}{y(c + 2\alpha b) + \alpha c + 2a + d + \alpha e}\left\{\ln(by^2 + y(e + c) + a + d + f) - \ln(by_0^2 + ey_0 + f)\right\}\right.$$

$$-\int_0^1 \frac{dy\,\alpha}{y(\alpha c + 2a) + d + \alpha e}\left\{\ln\left(ay^2 + dy + f\right) - \ln(by_0^2 + ey_0 + f)\right\}$$

$$\left.-\int_0^1 \frac{dy}{y(c + 2\alpha b + \alpha c + 2a) + d + \alpha c}\left\{\ln\left(y^2(a + b + c) + y(d + e) + f\right) - \ln(by_0^2 + ey_0 + f)\right\}\right]$$

(1

where we used the equation (109).

Let's make the substitutions: $y_1 = y_0 + \alpha$, $y_2 = -y_0/\alpha$ and $y_3 = y_0/(1-\alpha)$.

$$
\begin{aligned}
C \quad = \quad & i\frac{\pi^2}{2}\Bigg[\int_0^1 \frac{dy}{y(c+2\alpha b)+\alpha c+2a+d+\alpha e}\left\{\ln(by^2+y(e+c)+a+d+f)-\ln(by_1^2+(c+e)y_1+a+d+f)\right\} \\
& -\int_0^1 \frac{dy\,\alpha}{y(\alpha c+2a)+d+\alpha e}\left\{\ln\left(ay^2+dy+f\right)-\ln(ay_2^2+dy_2+f)\right\} \\
& -\int_0^1 \frac{dy(1-\alpha)}{y(c+2\alpha b+\alpha c+2a)+d+\alpha c}\left\{\ln\left(y^2(a+b+c)+y(d+e)+f\right)-\ln((a+b+c)y_3^2+(e+d)y_3+f)\right\}\Bigg],
\end{aligned}
\tag{130}
$$

Let's consider for example the second derivation. From the previous calculus we know that:

$$b{y_0}^2 + e y_0 + f - \frac{y_0}{\alpha} N(y_0) = a y_2^2 + d y_2 + f \tag{131}$$

but $N(y_0) = 0$, hence,

$$b{y_0}^2 + e y_0 + f = a y_2^2 + d y_2 + f \tag{132}$$

$$\begin{aligned} C \quad = \quad & i\frac{\pi^2}{2}\left[\frac{1}{(c+2\alpha b)}\int_0^1 \frac{dy}{y-y_{01}}\left\{\ln(by^2 + y(e+c) + a + d + f) - \ln(by_1^2 + (c+e)y_1 + a + d + f)\right\}\right. \\ & - \frac{\alpha}{\alpha c + 2a}\int_0^1 \frac{dy}{y-y_{02}}\left\{\ln\left(ay^2 + dy + f\right) - \ln(ay_2^2 + dy_2 + f)\right\} \\ & \left. - \frac{1-\alpha}{c + 2\alpha b + \alpha c + 2a}\int_0^1 \frac{dy}{y-y_{03}}\left\{\ln\left(y^2(a+b+c) + y(d+e) + f\right) - \ln((a+b+c)y_3^2 + (e+d)y_3 + f)\right\}\right], \\ = \quad & i\frac{\pi^2}{2}\left[S_3(y_{01}, y_{11}, y_{21}) - S_3(y_{02}, y_{12}, y_{22}) - S_3(y_{03}, y_{13}, y_{23})\right] \end{aligned} \tag{133}$$

where

$$y_{01} = -\frac{\alpha c + 2a + d + \alpha e}{c + 2\alpha b} \quad (134)$$

$$y_{02} = -\frac{d + \alpha e}{\alpha c + 2a} \quad (135)$$

$$y_{03} = -\frac{d + \alpha c}{c + 2\alpha b + \alpha c + 2a} \quad (136)$$

and $y_{11}$, and $y_{21}$ are the roots of

$$0 = by^2 + y(e + c) + a + d + f; \quad (137)$$

$y_{12}$, and $y_{22}$ are the roots of

$$0 = ay^2 + dy + f; \quad (138)$$

eventually, $y_{12}$, and $y_{22}$ are the roots of

$$0 = y^2(a + b + c) + y(d + e) + f. \quad (139)$$

$$R(y_0\,,\,y_1) = \int_0^1 dy\,\frac{1}{y\,-\,y_0}\,\Big\{\ln(y\,-\,y_1) - \ln(y_0\,-\,y_1)\Big\} \tag{140}$$

Assumption: $\mathrm{Im}(a(y\,-\,y_1)(y\,-\,y_2)$ has definite nonzero sign in the $y$-domain. It means that the argument of $\ln(y\,-\,y_1)$ never crosses the logarithm cut! Now we make the change of the variables $y\,\to\,y\,-\,y_1$, then

$$R = \int_{-y_1}^{1-y_1} dy\,\frac{1}{y\,+\,y_1\,-\,y_0}\,\Big\{\ln y\,-\,\ln(y_0\,-\,y_1)\Big\} \tag{141}$$

$$\begin{aligned} R(y_0\,,\,y_1) \quad = \quad & \ln\left(\frac{-y_0}{y_1\,-\,y_0}\right)\,\eta\left(-y_1\,,\,\frac{1}{y_0\,-\,y_1}\right) - \ln\left(\frac{1\,-\,y_0}{y_1\,-\,y_0}\right)\,\eta\left(1\,-\,y_1\,,\,\frac{1}{y_0\,-\,y_1}\right) \\ & + Sp\left(\frac{y_0}{y_0\,-\,y_1}\right) - Sp\left(\frac{1\,-\,y_0}{y_1\,-\,y_0}\right) \end{aligned} \tag{142}$$

Consider the integral

$$S_3(y_0, y_1, y_2) = \int_0^1 dy \frac{1}{y - y_0} \left[ \ln(ay^2 + by + c) - \ln(ay_0^2 + by_0 + c) \right] \tag{143}$$

Assumptions:

(a) $a$ is real;

(b) $b$, $c$, and $y_0$ may be complex;

(c) $\mathrm{Im}(ay^2 + by + c)$ has the same sign in the $y$ range, [0, 1]);

Let

$$ay^2 + by + c = a(y - y_1)(y - y_2), \quad y_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{144}$$

Notice that

$$a(y - y_1)(y - y_2) = ay^2 - ay(y_1 + y_2) + ay_1 y_2, \quad -(y_1 + y_2) = \frac{b}{a}, \quad y_1 y_2 = \frac{c}{a}. \tag{145}$$

Hence the imaginary part of the above expression is given by factors $-y_1 + y_2$ and $y_1 y_2$, for $y = 0$ the imaginary parts reads

$$a\mathrm{Im}(y_1 y_2)$$

while for $y$

$$-a\mathrm{Im}(y_1 + y_2) + a\mathrm{Im}(y_1 y_2)$$

The sign of $-\mathrm{Im}(y_1 + y_2) + \mathrm{Im}(y_1 y_2)$ must be the same as $\mathrm{Im}(y_1 y_2)$. The letter decides about sign of the expression.

The recruitment of the same sign of the imaginary part of the logarithm argument in the $y$-range implies that either

$$y\mathrm{Im}b + \mathrm{Im}c > 0, \quad \Rightarrow \quad \mathrm{Im}c > 0, \quad \mathrm{Im}c > -\mathrm{Im}b \tag{146}$$

or

$$y\mathrm{Im}b + \mathrm{Im}c < 0, \quad \Rightarrow \quad \mathrm{Im}c < 0, \quad \mathrm{Im}c < -\mathrm{Im}b. \tag{147}$$

Let introduce the infinitesimal parameters $\epsilon$ and $\delta$ which have opposite signs then $\mathrm{Im}(a(y - y_1)(y - y_2))$ and $\mathrm{Im}(a(y_0 - y_1)(y_0 - y_2))$ respectively.
Now

$$\ln(a(y - y_1)(y - y_2)) \to \ln((a - i\epsilon)(y - y_1)(y - y_2)) = \ln(a - i\epsilon) + \ln((y - y_1)(y - y_2)) \tag{148}$$

We applied the (1) rule for the case in which $a(y - y_1)(y - y_2)$ and $a - i\epsilon$ has the same signs of the imaginary parts. Analogically

$$\ln(a(y_0 - y_1)(y_0 - y_2)) \to \ln((a - i\delta)(y_0 - y_1)(y_0 - y_2)) = \ln(a - i\delta) + \ln((y_0 - y_1)(y_0 - y_2)) \tag{149}$$

We use above properties and we write

$$
\begin{aligned}
S_3 &= \int_0^1 dy \, \frac{1}{y - y_0} \left[ \ln((y - y_1)(y - y_2)) - \ln((y_0 - y_1)(y_0 - y_2)) + \ln(a - i\epsilon) - \ln(a - i\delta) \right] && (150) \\
&= \int_0^1 dy \, \frac{1}{y - y_0} \left[ \ln((y - y_1)(y - y_2)) - \ln((y_0 - y_1)(y_0 - y_2)) - \eta \left( a - i\epsilon, \frac{1}{a - i\delta} \right) \right] && (151) \\
&= \int_0^1 dy \, \frac{1}{y - y_0} \left[ \ln((y - y_1)(y - y_2)) - \ln((y_0 - y_1)(y_0 - y_2)) \right] - \eta \left( a - i\epsilon, \frac{1}{a - i\delta} \right) \int_0^1 dy \, \frac{1}{y - y_0} && (152)
\end{aligned}
$$

We easily perform the integral

$$
\int_0^1 dy \, \frac{1}{y - y_0}
$$

however, we notice that for $y_0 \in (0, 1)$ the result can not be uniquely obtained because we integrate along logarithm cut!

$$S_3 = \int_0^1 dy \frac{1}{y-y_0} \left[ \ln((y-y_1)(y-y_2)) - \ln((y_0-y_1)(y_0-y_2)) \right] - \eta\left(a - i\epsilon, \frac{1}{a-i\delta}\right) \ln\left(\frac{1-y_0}{y_0}\right) \quad (153)$$

$$S_3 = \int_0^1 dy \frac{1}{y-y_0} \left[ \ln(y-y_1) + \ln(y-y_2) + \eta(-y_1, -y_2) - \ln(y_0-y_1) - \ln(y_0-y_2) - \eta(y_0-y_1, y_0-y_2) \right]$$
$$-\eta\left(a - i\epsilon, \frac{1}{a-i\delta}\right) \ln\left(\frac{1-y_0}{y_0}\right) \quad (154)$$
$$= R(y_0, y_1) + R(y_0, y_2) + \left[ \eta(-y_1, -y_2) - \eta(y_0-y_1, y_0-y_2) - \eta\left(a - i\epsilon, \frac{1}{a-i\delta}\right) \right] \ln\left(\frac{1-y_0}{y_0}\right) \quad (155)$$

We have the difference in sign (red) with respect to t'Hooft Veltman paper, but it seems that it was misprint!
After imposing (142) we get

$$S_3(y_0, y_1, y_2) = Sp\left(\frac{y_0}{y_0-y_1}\right) - Sp\left(\frac{y_0-1}{y_0-y_1}\right) + Sp\left(\frac{y_0}{y_0-y_2}\right) - Sp\left(\frac{y_0-1}{y_0-y_2}\right)$$
$$+\ln\left(\frac{y_0}{y_0-y_1}\right) \eta(-y_1, 1/(y_0-y_1)) - \ln\left(\frac{y_0-1}{y_0-y_1}\right) \eta(1-y_1, 1/(y_0-y_1))$$
$$+\ln\left(\frac{y_0}{y_0-y_2}\right) \eta(-y_2, 1/(y_0-y_2)) - \ln\left(\frac{y_0-1}{y_0-y_2}\right) \eta(1-y_2, 1/(y_0-y_2))$$
$$+ \left[ \eta(-y_1, -y_2) - \eta(y_0-y_1, y_0-y_2) - \eta\left(a - i\epsilon, \frac{1}{a-i\delta}\right) \right] \ln\left(\frac{1-y_0}{y_0}\right) \quad (156)$$

$$D(p_1, p_2, p_3, p_4, m_1, m_2, m_3, m_4)$$
$$= \int \mathrm{d}_n q \frac{1}{(q^2+m_1^2)((q+p_1)^2+m_2^2)((q+p_1+p_2)^2+m_3^2)((q+p_1+p_2+p_3)^2+m_4^2)}$$

The final result is expressed in terms of 24-48 Spence functions, t'Hooft, Veltman, for not real masses it can be 108 Spence functions.

$$\frac{D}{i\pi^2}=\frac{A_1A_2A_3A_4}{k}$$

$$\times\Bigg[-\int_0^1 \mathrm{d}y \frac{1-\alpha}{(c+2\alpha b)(1-\alpha)y+d+e\alpha}\{\ln L_{24}(y)-\ln L_{24}(y_1)\}-R_{24}^1$$

$$-\int_0^1 \mathrm{d}y \frac{\alpha}{-(c+2\alpha\beta)\alpha y+d+e\alpha}\{\ln L_{34}(y)-\ln L_{34}(y_2)\}+R_{34}^2$$

$$+\int_0^1 \mathrm{d}y \frac{1}{(c+2\alpha b)y+d+e\alpha+c\alpha+2a}\{\ln L_{23}(y)-\ln L_{23}(y_3)\}+R_{23}^3$$

$$+\int_0^1 \mathrm{d}y \frac{1-\alpha}{(c+2\alpha b)(1-\alpha)y+d+(e+k)\alpha}\{\ln L_{14}(y)-\ln L_{14}(y_4)\}+R_{14}^4$$

$$+\int_0^1 \mathrm{d}y \frac{\alpha}{-(c+2\alpha b)\alpha y+d+(e+k)\alpha}\{\ln L_{34}(y)-\ln L_{34}(y_5)\}-R_{34}^5$$

$$-\int_0^1 \mathrm{d}y \frac{1}{(c+2\alpha b)y+d+(e+k)\alpha+c\alpha+2a}\{\ln L_{13}(y)-\ln L_{13}(y_6)\}-R_{13}^6$$

$$+\theta(-A_1A_2)S\Bigg], \qquad (6.12)$$

$$L_{ij}(y)=(-l_{ij}A_iA_j+m_i^2A_i^2+m_j^2A_j^2)y^2+(l_{ij}A_iA_j-2m_j^2A_j^2)y+m_j^2A_j^2-i\varepsilon\,. \qquad (6.13)$$

The total electron photon vertex reads

$$-ie\Gamma^{\mu} = -ie\gamma^{\mu} - ie\delta\Gamma^{\mu}, \quad \delta\Gamma^{\mu} \sim e^3 \tag{157}$$

$$\bar{u}(p')\left[-ie\delta\Gamma^{\mu}\right]u(p) = \underbrace{-e^3}_{=-ie(-ie^2)}\int\frac{d^4}{(2\pi)^n}\frac{\bar{u}(p')\left[\gamma^{\alpha}(\hat{p'}+\hat{l}+m)\gamma^{\mu}(\hat{p}+\hat{l}+m)\gamma_{\alpha}\right]u(p)}{((p'+l)^2-m^2+i\epsilon)((p+l)^2-m^2+i\epsilon)(l^2+i\epsilon)} \tag{158}$$

$$V^{\mu} = \bar{u}(p')\delta\Gamma^{\mu}u(p) = -ie^2\int\frac{d^4l}{(2\pi)^n}\frac{\bar{u}(p')\left[\gamma^{\alpha}(\hat{p'}+\hat{l}+m)\gamma^{\mu}(\hat{p}+\hat{l}+m)\gamma_{\alpha}\right]u(p)}{((p'+l)^2-m^2+i\epsilon)((p+l)^2-m^2+i\epsilon)(l^2+i\epsilon)} \tag{159}$$

Let's introduce

$$D \equiv ((p'+l)^2-m^2+i\epsilon)((p+l)^2-m^2+i\epsilon)(l^2+i\epsilon) \tag{160}$$

$$N \equiv \bar{u}(p')\left[\gamma^{\alpha}(\hat{p'}+\hat{l}+m)\gamma^{\mu}(\hat{p}+\hat{l}+m)\gamma_{\alpha}\right]u(p). \tag{161}$$

$$\int \frac{d^n l}{(2\pi)^n} \frac{N}{D} = 2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(1-x-y-z) \int \frac{d^n l}{(2\pi)^n} \frac{N}{D'}, \tag{162}$$

where $D' = D(l \to l')$

$$D' = [x((p'+l)^2 - m^2 + i\epsilon) + y((p+l)^2 - m^2 + i\epsilon) + z(l^2 + i\epsilon)]^3 = [l^2 + 2xp' \cdot l + 2yp \cdot l + (i\epsilon)]^3 \tag{163}$$

$$= [l'^2 - (xp' + yp)^2 + i\epsilon]^3 \tag{164}$$

$$l' = l + xp' + yp \tag{165}$$

$$D' = [l'^2 - \Delta + i\epsilon]^3 \tag{166}$$

$$\Delta = (x+y)^2 m_e^2 - xyq^2 \tag{167}$$

we have used the property that

$$2p' \cdot p = 2m_e^2 - q^2.$$

We remark that for most of physical examples $q^2 < 0$, hence $\Delta > 0$.
Eventually we write

$$\int \frac{d^n l}{(2\pi)^n} \frac{N(l)}{D} = 2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^n l'}{(2\pi)^n} \frac{N(l' = l - xp' - yp)}{[l'^2 - \Delta + i\epsilon]^3}, \tag{168}$$

In order to regularize the IR divergencies we introduce the photon mass $\mu$, in the following way:

$$\frac{1}{k^2} \to \frac{1}{k^2 - \mu^2 + i\epsilon} \tag{169}$$

then

$$\Delta = (x+y)^2 m_e^2 - xyq^2 + (1-x-y)\mu^2 \tag{170}$$

In order to simplify the denominator we use the properties

$$\overline{u}(p')\hat{p} = \overline{u}(p')m, \quad \hat{p}u(p) = m\,u(p), \quad \gamma^{\mu}\gamma^{\nu} = 2g^{\mu\nu} - \gamma^{\nu}\gamma^{\nu}. \tag{171}$$

Notice that

$$\overline{u}(p')\gamma^{\alpha}(\hat{p'} + \hat{l} + m) = \overline{u}(p')\,(2p'^{\alpha} - \hat{p}\gamma^{\alpha} + \gamma^{\alpha}\hat{l} + m\gamma^{\alpha}) = \overline{u}(p')(2p'^{\alpha} + \gamma^{\alpha}\hat{l}) \tag{172}$$

$$(\hat{p} + \hat{l} + m)\gamma_{\alpha}u(p) = (2p_{\alpha} - \gamma_{\alpha}\hat{p} + m\gamma_{\alpha} + \hat{l}\gamma_{\alpha})u(p) = (2p_{\alpha} + \hat{l}\gamma_{\alpha})u(p) \tag{173}$$

Hence

$$N = \overline{u}(p')(2p'^{\alpha} + \gamma^{\alpha}\hat{l})\gamma^{\mu}(2p_{\alpha} + \hat{l}\gamma_{\alpha})u(p)\overline{u}(p')\left[4\gamma^{\mu}p \cdot p' + 2\gamma^{\mu}\hat{l}\hat{p'} + 2\hat{p}\hat{l}\gamma^{\mu} + \gamma^{\alpha}\hat{l}\gamma^{\mu}\hat{l}\gamma_{\alpha}\right]u(p) \tag{174}$$

Now notice that:

$$\gamma^{\mu}\hat{l}\hat{p'} = 2\gamma^{\mu}(p' \cdot l) - 2p^{\mu}\hat{l} + \hat{p'}\gamma^{\mu}\hat{l} \tag{175}$$

$$\hat{p}\hat{l}\gamma^{\mu} = 2\gamma^{\mu}(p \cdot l) - 2\hat{l}p^{\mu} + \hat{l}\gamma^{\mu}\hat{p} \tag{176}$$

$$\gamma^{\alpha}\hat{l}\gamma^{\mu}\hat{l}\gamma_{\alpha} = 2l^{2}\gamma^{\mu} - 4l^{\mu}\hat{l} \tag{177}$$

$$N = \overline{u}(p')\left[4\gamma^{\mu}\left[p\cdot p' + (p'+p)\cdot l + (n-2)l^2\right] - 4(p^{\mu}+p'^{\mu})\hat{l} + (4m_e - 2(n-2)\hat{l})l^{\mu}\right]u(p) \quad (178)$$

where n is the dimension of the integral. In this stage of calculus we obtained the same expression for nominator as Pokorski in his textbook. We make the substitution $l = l' - xp' - yp$ in the numerator, but before we do it, let's look at the expression $2(n-2)\hat{l}l^{\mu}$ which under the integral over $l'$ changes to

$$\overline{u}(p)2(n-2)\hat{l}l^{\mu}u(p) \rightarrow \overline{u}(p)2(n-2)\left[\gamma^{\alpha}l_{\alpha}l^{\mu} + m_e(x+y)(xp'^{\mu}+yp^{\mu})\right]u(p) \quad (179)$$

$$\rightarrow \overline{u}(p)2(n-2)\left[\gamma^{\alpha}l_{\alpha}l^{\mu} + m_e(x+y)\left((x+y)\frac{(p'^{\mu}+p^{\mu})}{2} + (x-y)\frac{q^{\mu}}{2}\right)\right]u(p) \quad (180)$$

$$\rightarrow \overline{u}(p)2(n-2)\left[\gamma^{\mu}\frac{l^2}{n} + m_e(x+y)^2\frac{(p'^{\mu}+p^{\mu})}{2}\right]u(p) \quad (181)$$

Red contribution will give $\gamma^{\mu}\mathcal{I}_2$, hence $l^{\mu}l^{\mu} \rightarrow l^2/n$, the blue contribution is antisymmetric under $x \leftrightarrow y$ exchange, while the integral should not depend on this exchange, hence it must vanish!
Notice that after the $l \rightarrow l'$ change and applying the so-called Gordon decomposition

$$(p^{\mu}+p'^{\mu})\overline{u}(p')u(p) = 2m_e\overline{u}(p')\gamma^{\mu}u(p) - i\overline{u}(p')\sigma^{\mu\nu}q_{\nu}u(p) \quad (182)$$

we get

$$N = f_1(q^2,x,y)\overline{u}(p')\gamma^{\mu}u(p) + f_2(q^2,x,y)\overline{u}(p')\frac{i\sigma^{\mu\nu}q^{\nu}}{2m_e}u(p), \quad (183)$$

where

$$f_1(q^2,x,y) = 4m_e^2(1-x-y) - 4(1-x-y)\frac{q^2}{2} + (n-2)\left(l^2\frac{n-2}{n} - (x+y)^2m_e^2 - xyq^2\right) \quad (184)$$

$$f_2(q^2,x,y) = -4\left(1 - \frac{(n-2)}{2}(x+y)\right)(x+y)m_e^2 \quad (185)$$

$$V^{\mu} = F_1(q^2)\overline{u}(p')\gamma^{\mu}u(p) + F_2(q^2)\overline{u}(p')\frac{i\sigma^{\mu\nu}q^{\nu}}{2m_e}u(p), \tag{186}$$

where

$$F_1(q^2) = -2ie^2\int_0^1 dx\int_0^{1-x} dy f_1(q^2,x,y) \tag{187}$$

$$F_2(q^2) = -2ie^2\int_0^1 dx\int_0^{1-x} dy f_2(q^2,x,y) \tag{188}$$

Now we see that it seems to be more operative to use $z = x + y$ variable and $x$ instead of $x$ and $y = z - x$.

$$F_1(q^2) = -2ie^2\int_0^1 dz\int_0^z dx\left[4(1-z)\left(m_e^2-\frac{q^2}{2}\right)\mathcal{I}_0(n,3)\right.$$
$$\left.+(n-2)\left(\frac{n-2}{n}\mathcal{I}_0(n,3)-\left(z^2m_e^2+x(z-x)q^2\right)\mathcal{I}_0(n,3)\right)\right] \tag{189}$$

$$F_2(q^2) = 8ie^2\int_0^1 dz\int_0^z dx\left(1-\frac{(n-2)}{2}z\right)zm_e^2\mathcal{I}_0(n,3) \tag{190}$$

Only contribution is ultraviolet divergent. In the rest of them we set $n = 4$.

$$F_1(q^2) = -\frac{e^2}{8\pi^2}\left[\left(2m_e^2 - q^2\right) A_0 - m_e^2 A_2 - q^2 A_3\right] + \frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon_{UV}} - \gamma - 2 + \ln(\pi) + 2\ln(2) - 2A_4(q^2)\right) \tag{191}$$

$$F_2(q^2) = 8ie^2 m_e^2 \int_0^1 dz \int_0^z dx\,(1-z)\,z\mathcal{I}_0(4,3) = \frac{e^2}{4\pi^2} m_e^2 A_1(q^2) \tag{192}$$

$$\mathcal{I}_0(4-\epsilon_{UV},2) = i(2\sqrt{\pi})^{-(4-\epsilon_{UV})}\Gamma(\epsilon_{UV}/2)\Delta^{-\epsilon_{UV}/2} \tag{193}$$

$$\approx \frac{i}{16\pi^2}\left(\frac{2}{\epsilon_{UV}} - \gamma - \ln\Delta + 2\ln(2\sqrt{\pi})\right) \tag{194}$$

$$\mathcal{I}_0(4,3) = -\frac{i}{32\pi^2}\Delta^{-1} \tag{195}$$

$$= \frac{i}{16\pi^2}\left(\frac{2}{\epsilon_{UV}} - \ln(\Delta) - \gamma - 2 + \ln(\pi) + 2\ln(2)\right) \tag{196}$$

$$A_0(q^2) = \int_0^1 dz \int_0^z dx \frac{1-z}{z^2 m_e^2 - x(z-x)q^2 + (1-z)\mu^2} \tag{197}$$

$$A_1(q^2) = \int_0^1 dz \int_0^z dx \frac{(1-z)z}{z^2 m_e^2 - x(z-x)q^2 + (1-z)\mu^2} \tag{198}$$

$$A_2(q^2) = \int_0^1 dz \int_0^z dx \frac{z^2}{z^2 m_e^2 - x(z-x)q^2 + (1-z)\mu^2} \tag{199}$$

$$A_3(q^2) = \int_0^1 dz \int_0^z dx \frac{x(z-x)}{z^2 m_e^2 - x(z-x)q^2 + (1-z)\mu^2} \tag{200}$$

$$A_4(q^2) = \int_0^1 dz \int_0^z dx \ln(z^2 m_e^2 - x(z-x)q^2 + (1-z)\mu^2) \tag{201}$$

Notice that $A_1$ integral is not divergent when $\mu \to 0$, analogically the integral $A_2$

$$A_0(0) = \int_0^1 dz \frac{(1-z)z}{z^2 m_e^2 + (1-z)\mu^2} \tag{202}$$

$$A_1(0) = \int_0^1 dz \frac{(1-z)z^2}{z^2 m_e^2 + (1-z)\mu^2} \tag{203}$$

$$A_2(0) = \int_0^1 dz \frac{z^3}{z^2 m_e^2 + (1-z)\mu^2} \tag{204}$$

$$A_3(0) = \frac{1}{6}\int_0^1 dz \frac{z^3}{z^2 m_e^2 + (1-z)\mu^2} \tag{205}$$

$$A_4(0) = \int_0^1 dz z \ln(z^2 m_e^2 + (1-z)\mu^2) \tag{206}$$

We see that only first integral is divergent.

$$A_1(0) = \int_0^1 dz \frac{(1-z)z^2}{z^2 m_e^2} = \int_0^1 dz \frac{(1-z)}{m_e^2} = \frac{1}{2m_e^2} \tag{207}$$

$$A_2(0) = \int_0^1 dz \frac{z^3}{z^2 m_e^2 + (1-z)\mu^2} \to \frac{1}{2m_e^2} = \int_0^1 dz \frac{z}{m_e^2} = \frac{1}{2m_e^2} \tag{208}$$

$$A_3(0) = \frac{1}{6}\int_0^1 dz \frac{z^3}{z^2 m_e^2 + (1-z)\mu^2} = \frac{1}{6}\int_0^1 dz \frac{z}{m_e^2} = \frac{1}{12m_e^2} \tag{209}$$

$$A_4(0) = \int_0^1 dzz \ln(z^2 m_e^2 + (1-z)\mu^2) = \int_0^1 dzz \ln(z^2 m_e^2) = \frac{1}{2}\ln m_e^2 - \frac{1}{2} \tag{210}$$

Consider the integral $A_0(0)$

$$A_0(0) = \int_0^1 dz \frac{(1-z)z}{z^2 m_e^2 + (1-z)\mu^2} = \int_0^1 dz \frac{z}{z^2 m_e^2 + (1-z)\mu^2} - \frac{1}{m_e^2} \tag{211}$$

$$\tag{212}$$

Now we compute the integral

$$* = \frac{1}{m_e^2(z_2 - z_1)} \left[ z_2 \ln(1-z_2) - z_2 \ln(-z_2) - z_1 \ln(z - z_1) + z_1 \ln(-z_1) \right] \tag{213}$$

where

$$z_{1,2} = \frac{-\mu^2 \pm \sqrt{\mu^4 - 4\mu^2 m_e^2}}{2m_e^2} \quad \sqrt{\mu^4 - 4\mu^2 m_e^2} = 2i\mu m_e \sqrt{1 - \frac{\mu^2}{4m_e^2}} \approx 2i\mu m_e - i\frac{\mu^3}{4m_e} \approx 2im\mu \tag{214}$$

Hence

$$z_{1,2} \approx \mp i\frac{\mu}{m_e}, \, z_2 - z_1 \approx 2i\frac{\mu}{m_e} \approx 2z_2 = -2z_1 \tag{215}$$

Hence

$$* \approx -\frac{1}{m_e^2(z_2 - z_1)}\left[z_2 \ln(-z_2) - z_1 \ln(-z_1)\right] \quad (216)$$

$$\approx -\frac{1}{2m_e^2}\left[\ln\left(-i\frac{\mu}{m_e}\right) + \ln\left(i\frac{\mu}{m_e}\right)\right] \quad (217)$$

$$= -\frac{1}{m_e^2}\ln\left(\frac{\mu}{m_e}\right) \quad (218)$$

Hence

$$A_0 \approx -\frac{1}{m_e^2}\left(\ln\left(\frac{\mu}{m_e}\right) + 1\right) \quad (219)$$

We can easily compute the $F_2(q^2 = 0)$, which is not ultraviolet divergent, and not photon mass dependent!

$$F_2(0) = \frac{e^2}{4\pi^2} m_e^2 A_1(0) = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi} \quad (220)$$

which is the same result as obtained by Schwinger in 1949!

We need to evaluate the integrals of the form

$$
\begin{aligned}
A_0(q^2) &= S - S_1 && (221)\\
A_1(q^2) &= S_1 - S_2 && (222)\\
A_2(q^2) &= S_2 && (223)\\
A_3(q^2) &= -\frac{1}{2q^2} + \frac{m_e^2}{q^2} S_2 + \frac{\mu^2}{q^2} S && (224)\\
A_4(q^2) &= S_L\,, && (225)
\end{aligned}
$$

where integrals $S$, $S_1$, $S_2$ and $S_L$ are defined in the appendix.

$$F_1(q^2) = -\frac{e^2}{8\pi^2}\left[\left(2m_e^2 - q^2 - \mu^2\right)(S - S_1) - 2m_e^2 S_2 + \frac{1}{2}\right] + \frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon_{UV}} - \gamma - 2 + \ln(\pi) + 2\ln(2) - 2S_L\right) \tag{226}$$

$$F_2(q^2) = \frac{e^2}{4\pi^2} m_e^2 (S_1 - S_2) \tag{227}$$

$$S = \int_0^1 dz \int_0^z dx \frac{1}{m^2 z^2 - x(z-x)q^2 + (1-z)\mu^2} = \frac{1}{q^2(1-2a)} \left[ S_3(y_1, y_-^a, y_+^a) - S_3(y_2, y_-^a, y_+^a) + S_3(y_3, y_-^b, y_+^b) \right] \tag{228}$$

$y_+^a$ and $y_-^a$ roots of $0 = m^2 y'^2 - \mu^2 y' + \mu^2$, $y_+^b$ and $y_-^b$ roots of $q^2 y'^2 - q^2 y' + m^2 = 0$.

$$S_1 = \int_0^1 dz \int_0^z dx \frac{z}{m^2 z^2 - x(z-x)q^2 + (1-z)\mu^2} = -\frac{2}{q^2(2a-1)} \ln\left(\frac{a}{a-1}\right) \tag{229}$$

$$S_2 = \int_0^1 dz \int_0^z dx \frac{z^2}{m^2 z^2 - x(z-x)q^2 + (1-z)\mu^2} = -\frac{1}{(2a-1)q^2} \ln \frac{a}{a-1} \tag{230}$$

$$S_L = \int_0^1 dz \int_0^z dx \ln(m^2 z^2 - x(z-x)q^2 + (1-z)\mu^2) = \frac{1}{2}\ln(m^2) - \frac{3}{2} - \frac{(1-2a)}{2} \ln \frac{a}{a-1} \tag{231}$$

Now we take $a$ which is one of the root of equation

$$0 = m^2 - aq^2 + a^2 q^2 . \tag{232}$$

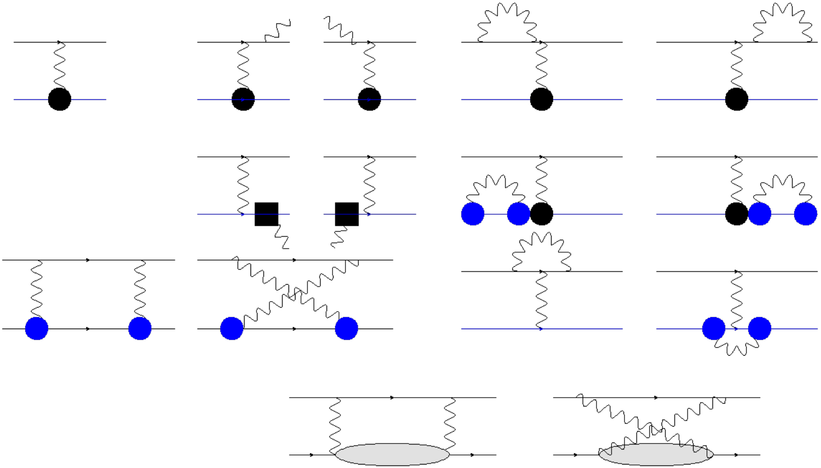
$$a_\pm = \frac{q^2 \pm \sqrt{q^4 - 4q^2 m^2}}{2q^2} \tag{233}$$

notice that $a_+ < 0$, while $a_- > 0$ and $a_- > 1$ we choose for further computation $a_-$!

- FeynCalc is a Mathematica package for algebraic calculations in elementary particle physics.
  Some of the features of FeynCalc are:
  - Passarino-Veltman reduction of one-loop amplitudes to standard scalar integrals
  - Tools for frequently occuring tasks like Lorentz index contraction, color factor calculation, Dirac matrix manipulation and traces, etc.
  - Tensor and Dirac algebra manipulations (including traces) in 4 or D dimensions
  - Generation of Feynman rules from a lagrangian
  - Tools for non-commutative algebra
  - SU(N) algebra
  - Tables of integrals, convolutions and Feynman rules
  - Special convolution, Mellin transform and other integral tables
  - Tools for calculating 2-loop propagator-type diagrams
  - FORM and FORTRAN code generation
  - Translation to and from FORM
- FeynCalc is maintained and developed by Rolf Mertig and Frederik Orellana.

see: *http://www.feyncalc.org/*

- LoopTools is a package for evaluation of scalar and tensor one-loop integrals based on the FF package by G.J. van Oldenborgh. It features an easy Fortran, C++, and Mathematica interface to the scalar one-loop functions of FF and in addition provides the 2-, 3-, and 4-point tensor coefficient functions.
- LoopTools has been published in Comput. Phys. Commun. 118 (1999) 153 [hep-ph/9807565]. FF has been published in Z. Phys. C46 (1990) 425 [scanned version from KEK].



$$\sigma = \sigma_{1PE}(\alpha^2) + \underbrace{\sigma_{Bremsstrahlung}(\alpha^3)}_{Added\ Incoherently} + \sigma_{1PE\times 2PE}(\alpha^3) + \sigma_{2PE}(\alpha^4) + \underbrace{\sigma_{2\times Bremsstrahlung}(\alpha^4)}_{Added\ Incoherently} + \ldots \quad (234)$$

- off-shell black and blue vertices proton-$\gamma$-proton, Compton Scattering;
- intermediate state for TPE diagrams?
- How to compute, assuming some form-factors, usually Soft-Photon approximation;
- Y. S. Tsai, PR 122, 1898 (1961) (soft-photon approximation, proton structure neglected);
- L. C. Maximon, J. A. Tjon, PRC62, 054320 (2000) (soft photon approximation but $k = 0$ and $k = q$);
- Blunden, Tjon, Mielnitchouk, PRL, 142304 (2003); Tjon, Blunden, Mielnitchouk, PRC79, 055201 (2009) (**TPE**, $N$, $\Delta$);
- Parity Violating e-p scattering: Tjon, Mielnitchouk, PRL 100, 082003, (2008); Zhou, Kao, Yang, PRL99, 262001 (2007);
- C. E. Carlson, M. Vanderhaeghen, *Constraining off-shell effects using low-energy Compton scattering*, arXiv:1109.3779 (**TPE** and Compton Scattering)

Bremsstrahlung correction is caused by emission of the soft photons.

$$e^{-}(k) + p(p) \rightarrow e^{-}(k') + p(p') + \gamma(l). \tag{235}$$

$$Bremsstrahlung = \overline{u}(p')(ie)\Gamma^{\mu}_{\gamma}(l)\epsilon^{*}_{\mu}(l)\frac{i(\hat{p'} + \hat{l} + M)}{(p' + l)^2 - M^2 + i\epsilon} \times (similarly\ as\ at\ \mathcal{M}_0) \tag{236}$$

Assume the simplest scenario:

$$\Gamma^{\mu}_{\gamma}(l) \approx \Gamma^{\mu}_{p}(l) \tag{237}$$

Then we must evaluate

$$\overline{u}(p')\Gamma^{\mu}_{\gamma}(l)\epsilon^{*}_{\mu}(l)(\hat{p'} + \hat{l} + M) \approx \overline{u}(p')\Gamma^{\mu}_{\gamma}(l)\epsilon^{*}_{\mu}(l)(\hat{p'} + M) = \overline{u}(p')2p'^{\mu}F_1(l^2 = 0) \tag{238}$$

Hence, we multiply the scattering amplitude by

$$\frac{2p'^{\mu}F_1(0)}{l^2 + 2p' \cdot l + i\epsilon} = \frac{2p'^{\mu}F_1(0)}{2p' \cdot l + i\epsilon} \rightarrow \frac{p'^{\mu}F_1(0)}{p' \cdot l + i\epsilon} = \frac{p'^{\mu}F_1(0)}{E'_p|\mathbf{l}| - \mathbf{l}\cdot\mathbf{p}' + i\epsilon} \tag{239}$$

In practise the proton's Bremsstrahlung amplitude reads

$$\mathcal{M}(Proton's\ Bremsstrahlung) \approx \mathcal{M}_0 \frac{ie\epsilon^{*}_{\mu}p'^{\mu}F_1(0)}{p' \cdot l + i\epsilon} \tag{240}$$

Analogically we consider lepton soft emission, but the $p'$ four-momentum is replaced with $k'$. For this case the scattering amplitude reads

$$\mathcal{M}(electron's\ Bremsstrahlung) \approx \mathcal{M}_0 \frac{-ie\epsilon^{*}_{\mu}k'^{\mu}}{k' \cdot l + i\epsilon} \tag{241}$$

Hence the differential cross section for the emission of the soft proton photon reads

$$\frac{d\sigma}{dQ^2}_{Bremsstrahlung} \approx \frac{d\sigma}{dQ^2}\bigg|_{Born}\frac{e^2}{(2\pi)^3}\int\frac{d^3l}{2|\mathbf{l}|}\left|\frac{\epsilon^{*}_{\mu}p'^{\mu}F_1(0)}{p' \cdot l + i\epsilon} + \frac{\epsilon^{*}_{\mu}p^{\mu}F_1(0)}{p \cdot l + i\epsilon} + \frac{\epsilon^{*}_{\mu}k'^{\mu}}{k' \cdot l + i\epsilon} + \frac{\epsilon^{*}_{\mu}k^{\mu}}{k \cdot l + i\epsilon}\right|^2 \tag{242}$$

Integral

$$Br = \frac{1}{(2\pi)^3} \int_{|\mathbf{l}|<\omega} \frac{d^3 l}{2E_l} \frac{1}{(k' \cdot l)(p' \cdot l)}, \tag{243}$$

of interests, where $E_l = \sqrt{\mathbf{l}^2 + \lambda^2}$, $\lambda$ is the photon mass.
It is divergent in the photon mass and depends on $\omega$! Detailed analytical result can be found in G. 't Hooft anf M. Veltman, Nucl. Phys. **B**153 (1979) 365.

$$i\mathcal{M}_a^{(2)} = e^4 \int \frac{d^4 l}{(2\pi)^4} \frac{\overline{u}(k')\gamma^\mu(\hat{k'} - \hat{l})\gamma^\nu u(k)\overline{u}(p')\Gamma_\mu(-l)(\hat{p'} + \hat{l} + M)\Gamma_\nu(q+l)u(p)}{[(q+l)^2 + i\epsilon][l^2 + i\epsilon][(k'-l)^2 - m_e^2 + i\epsilon][(p'+l)^2 - M^2 + i\epsilon]} \quad (244)$$

$$i\mathcal{M}_b^{(2)} = e^4 \int \frac{d^4 l}{(2\pi)^4} \frac{\overline{u}(k')\gamma^\mu(\hat{k'} - \hat{l})\gamma^\nu u(k)\overline{u}(p')\Gamma_\nu(q+l)(\hat{p} - \hat{l} + M)\Gamma_\mu(-l)u(p)}{\left[(q+l)^2 + i\epsilon\right]\left[l^2 + i\epsilon\right][(k'-l)^2 - m_e^2 + i\epsilon][(p-l)^2 - M^2 + i\epsilon]} \quad (245)$$

**Soft Photon Approximation**, main contribution from $l = 0$ pole (As done in PRL ?? Mielnitchouk...)

$$i\mathcal{M}_a^{(2)}(l \to 0) \to e^4 \int \frac{d^4 l}{(2\pi)^4} \frac{\overline{u}(k')\gamma^\mu \hat{k'}\gamma^\nu u(k)\overline{u}(p')\Gamma_\mu(0)(\hat{p'} + M)\Gamma_\nu(q)u(p)}{[(q+l)^2 + i\epsilon][l^2 + i\epsilon][(k'-l)^2 - m_e^2 + i\epsilon][(p'+l)^2 - M^2 + i\epsilon]} \quad (246)$$

$$i\mathcal{M}_b^{(2)}(l \to 0) \to e^4 \int \frac{d^4 l}{(2\pi)^4} \frac{\overline{u}(k')\gamma^\mu \hat{k'}\gamma^\nu u(k)\overline{u}(p')\Gamma_\nu(q)(\hat{p} + M)\Gamma_\mu(0)u(p)}{\left[(q+l)^2 + i\epsilon\right]\left[l^2 + i\epsilon\right][(k'-l)^2 - m_e^2 + i\epsilon][(p-l)^2 - M^2 + i\epsilon]} \quad (247)$$

$$i\mathcal{M}_a^{(2)}(l + q \to 0) = e^4 \int \frac{d^4 l}{(2\pi)^4} \frac{\overline{u}(k')\gamma^\mu \hat{k}\gamma^\nu u(k)\overline{u}(p')\Gamma_\mu(q)(\hat{p} + M)\Gamma_\nu(0)u(p)}{[(q+l)^2 + i\epsilon][l^2 + i\epsilon][(k'-l)^2 - m_e^2 + i\epsilon][(p'+l)^2 - M^2 + i\epsilon]} \quad (248)$$

$$i\mathcal{M}_b^{(2)}(l + q \to 0) = e^4 \int \frac{d^4 l}{(2\pi)^4} \frac{\overline{u}(k')\gamma^\mu \hat{k}\gamma^\nu u(k)\overline{u}(p')\Gamma_\nu(0)(\hat{p'} + M)\Gamma_\mu(q)u(p)}{\left[(q+l)^2 + i\epsilon\right]\left[l^2 + i\epsilon\right][(k'-l)^2 - m_e^2 + i\epsilon][(p-l)^2 - M^2 + i\epsilon]} \quad (249)$$

$$
\begin{aligned}
i\mathcal{M}_a^{(2)}(l+q\to 0) &= \frac{ie^4}{16\pi^2}\overline{u}(k',s')\gamma^\mu\hat{k}\gamma^\nu u(k,s)\overline{u}(p',r')\Gamma_\mu(q)(\hat{p}+M)\Gamma_\nu(0)u(p,r) \\
&\quad D_0(m^2,-2k'\cdot p+m^2+M^2,2p\cdot q+M^2+q^2,q^2,M^2,2k'\cdot q+m^2+q^2,0,m^2,M^2,0) \qquad (250)\\
i\mathcal{M}_b^{(2)}(l+q\to 0) &= \frac{ie^4}{16\pi^2}\overline{u}(k',s')\gamma^\mu\hat{k}\gamma^\nu u(k,s)\overline{u}(p',r')\Gamma_\nu(0)(\hat{p'}+M)\Gamma_\mu(q)u(p,r) \\
&\quad D_0(m^2,-2k'\cdot p'+m^2+M^2,2p'\cdot q+M^2+q^2,q^2,M^2,2k'\cdot q+m^2+q^2,0,m^2,M^2 \qquad (251)
\end{aligned}
$$

## Try to compute exactly

$$\Gamma_p^\mu = \gamma^\mu F_1^p + \frac{i\sigma^{\mu\nu} q_\nu}{2M} F_2^p \tag{252}$$

$$F_1^p(q^2) = -\frac{4M_p^2}{q^2 - 4M_p^2}\left[G_E^p(q^2) - \frac{q^2}{4M_p^2} G_M^p(q^2)\right] \tag{253}$$

$$F_2^p(q^2) = -\frac{4M_p^2}{q^2 - 4M_p^2}\left[G_M^p(q^2) - G_E^p(q^2)\right] \tag{254}$$

where

$$G_E(q^2) = \sum_k \frac{\mathcal{C}_k^E}{(q^2 - P_k^E)^{n_k^E}}, \quad G_M(q^2)/\mu_p = \sum_k \frac{\mathcal{C}_k^M}{(q^2 - P_k^M)^{n_k^M}} \tag{255}$$

Notice that

$$\frac{1}{(l^2 - A)(l^2 - B)} = \frac{1}{B - A}\left(\frac{1}{(l^2 - B)} - \frac{1}{(l^2 - A)}\right), \quad \text{for } A \neq B. \tag{256}$$

$$\frac{1}{(q^2 - P_k^E)^{n_k^E}} = -\frac{(-1)^{n_k^E}}{(n_k^E - 1)!}\left(\frac{d}{dt^2}\right)^{n_k^E - 1} \frac{1}{q^2 - t^2}\Bigg|_{t = P_k^E} \tag{257}$$

$$i\mathcal{M}^{(2)} \sim \sum_{\alpha,\beta} C_{\alpha,\beta} \frac{\partial^{n_\alpha}}{\partial M_\alpha^2} \frac{\partial^{n_\beta}}{\partial M_\beta^2} \int \frac{d^4 l}{(2\pi)^4} \underbrace{\frac{Al^2 + Bl \cdot v + C}{[l^2 - M_\alpha^2][(q+l)^2 - M_\beta^2][(q+l)^2 + i\epsilon][l^2 + i\epsilon][(k'-l)^2 - m_e^2 + i\epsilon][(p'+l)^2 - M^2 + i\epsilon]}}_{*} \quad (258)$$

$$\begin{aligned} *(C) \quad = \quad & \frac{i}{8\pi^2} \Bigg( -\frac{1}{M_\alpha^2 M_\beta^2} D_0 \left( m^2, 2k' \cdot q + m^2 + q^2, M^2 - 2p \cdot q + q^2, M^2, q^2, 2k' \cdot p + M^2 + m^2, M_\alpha^2, m^2, 0, M^2 \right) \\ & + \frac{1}{M_\alpha^2 M_\beta^2} D_0 \left( 2k' \cdot p + M^2 + m^2, m^2, q^2, M^2 - 2p \cdot q + q^2, M^2, 2k' \cdot q + m^2 + q^2, M^2, m^2, 0, 0 \right) \\ & - \frac{1}{M_\alpha^2 M_\beta^2} D_0 \left( 2k' \cdot q + m^2 + q^2, m^2, M^2, M^2 - 2p \cdot q + q^2, q^2, 2k' \cdot p + M^2 + m^2, M_\beta^2, m^2, 0, M^2 \right) \\ & + \frac{1}{M_\alpha^2 M_\beta^2} D_0 \left( 2k' \cdot q + m^2 + q^2, 2k' \cdot p + M^2 + m^2, M^2, q^2, M^2 - 2p \cdot q + q^2, m^2, M_\beta^2, m^2, M^2, M_\alpha^2 \right) \Bigg) \quad (259) \end{aligned}$$

Useful observation $v$ is either $q$ or $k'$ or $p'$! It allows to simplify, however, the full result contains hundreds of scalar loop integrals.

$$*(A) \quad = \quad \frac{i}{8\pi^2}\left(-\frac{D_0\left(m^2, M^2, M^2-2p\cdot q+q^2, m^2+2k'\cdot q+q^2, M^2+m^2+2k'\cdot p, q^2, m^2, M_\alpha^2, M^2, 0\right)}{M_\beta^2}\right.$$

$$+\frac{D_0\left(m^2, M^2, M^2-2p\cdot q+q^2, m^2+2k'\cdot q+q^2, M^2+m^2+2k'\cdot p, q^2, m^2, M_\alpha^2, M^2, M_\beta^2\right)}{M_\beta^2}$$

$$-\frac{D_0\left(m^2, M^2+m^2+2k'\cdot p, M^2-2p\cdot q+q^2, q^2, M^2, m^2+2k'\cdot q+q^2, 0, m^2, M^2, 0\right)q^2}{M_\alpha^2 M_\beta^2}$$

$$+\frac{D_0\left(m^2, M^2+m^2+2k'\cdot p, M^2-2p\cdot q+q^2, q^2, M^2, m^2+2k'\cdot q+q^2, 0, m^2, M^2, M_\beta^2\right)q^2}{M_\alpha^2 M_\beta^2}$$

$$+\frac{D_0\left(m^2, M^2+m^2+2k'\cdot p, M^2-2p\cdot q+q^2, q^2, M^2, m^2+2k'\cdot q+q^2, M_\alpha^2, m^2, M^2, 0\right)q^2}{M_\alpha^2 M_\beta^2}$$

$$-\frac{D_0\left(m^2, M^2+m^2+2k'\cdot p, M^2-2p\cdot q+q^2, q^2, M^2, m^2+2k'\cdot q+q^2, M_\alpha^2, m^2, M^2, M_\beta^2\right)q^2}{M_\alpha^2 M_\beta^2}$$

$$+\frac{D_0\left(q^2, m^2, M^2+m^2+2k'\cdot p, M^2-2p\cdot q+q^2, m^2+2k'\cdot q+q^2, M^2, 0, 0, m^2, M^2\right)q^2}{M_\alpha^2 M_\beta^2}$$

$$-\frac{D_0\left(q^2, m^2, M^2+m^2+2k'\cdot p, M^2-2p\cdot q+q^2, m^2+2k'\cdot q+q^2, M^2, 0, M_\alpha^2, m^2, M^2\right)q^2}{M_\alpha^2 M_\beta^2}$$

$$-\frac{D_0\left(q^2, m^2, M^2+m^2+2k'\cdot p, M^2-2p\cdot q+q^2, m^2+2k'\cdot q+q^2, M^2, M_\beta^2, 0, m^2, M^2\right)q^2}{M_\alpha^2 M_\beta^2}$$

$$
\begin{aligned}
*(B, v = q) \quad = \quad & \frac{i}{8\pi^2} \left( -\frac{C_0\left(M^2, q^2, M^2 - 2p\cdot q + q^2, M^2, 0, 0\right)}{2M_\alpha^2 M_\beta^2} \right. \\
& + \frac{C_0\left(M^2, q^2, M^2 - 2p\cdot q + q^2, M^2, 0, M_\beta^2\right)}{2M_\alpha^2 M_\beta^2} + \frac{C_0\left(M^2, q^2, M^2 - 2p\cdot q + q^2, M^2, M_\alpha^2, 0\right)}{2M_\alpha^2 M_\beta^2} \\
& - \frac{C_0\left(M^2, q^2, M^2 - 2p\cdot q + q^2, M^2, M_\alpha^2, M_\beta^2\right)}{2M_\alpha^2 M_\beta^2} \\
& - \frac{D_0\left(m^2, M^2, M^2 - 2p\cdot q + q^2, m^2 + 2k'\cdot q + q^2, M^2 + m^2 + 2k'\cdot p, q^2, m^2, M_\alpha^2, M^2, 0\right)}{2M_\beta^2} \\
& + \frac{D_0\left(m^2, M^2, M^2 - 2p\cdot q + q^2, m^2 + 2k'\cdot q + q^2, M^2 + m^2 + 2k'\cdot p, q^2, m^2, M_\alpha^2, M^2, M_\beta^2\right)}{2M_\beta^2} \\
& + \frac{D_0\left(m^2, M^2 + m^2 + 2k'\cdot p, M^2 - 2p\cdot q + q^2, q^2, M^2, m^2 + 2k'\cdot q + q^2, 0, m^2, M^2, 0\right) k'\cdot q}{M_\alpha^2 M_\beta^2} \\
& - \frac{D_0\left(m^2, M^2 + m^2 + 2k'\cdot p, M^2 - 2p\cdot q + q^2, q^2, M^2, m^2 + 2k'\cdot q + q^2, 0, m^2, M^2, M_\beta^2\right) k'\cdot q}{M_\alpha^2 M_\beta^2} \\
& - \frac{D_0\left(m^2, M^2 + m^2 + 2k'\cdot p, M^2 - 2p\cdot q + q^2, q^2, M^2, m^2 + 2k'\cdot q + q^2, M_\alpha^2, m^2, M^2, 0\right) k'\cdot q}{M_\alpha^2 M_\beta^2} \\
& + \frac{D_0\left(m^2, M^2 + m^2 + 2k'\cdot p, M^2 - 2p\cdot q + q^2, q^2, M^2, m^2 + 2k'\cdot q + q^2, M_\alpha^2, m^2, M^2, M_\beta^2\right) k'\cdot q}{M_\alpha^2 M_\beta^2}
\end{aligned}
$$