Lattice Boltzmann Method

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Abstract
The flow of a fluid is described by Navier-Stokes equation[1]. Unfortunately, it can be solved analytically only for small group of problems. In this project I will use Multiple Relaxation Time Lattice Boltzmann method to reproduce von Karman vortex street.

1 Model description
Lattice Boltzmann method evolved out of Lattice Gas Automata, which simulated a gas through particles at discrete points in space, represented by boolean variables. LBM replaces the boolean variables of LGA with the discretized probability distribution functions $f$ which eliminates the need for ensemble averaging.

1.1 Lattice Boltzmann equation
Let us start with the continuous Boltzmann equation which describes time evolution of our hydrodynamic system,
$$\frac{\partial f}{\partial t} + \vec{e} \cdot \nabla f = \Omega_k,$$
where $f$ is distribution function, $\vec{e}$ is velocity and $\Omega_k$ is collision operator. In simplest case $\Omega_k$ is linear operator in BGK approximation:
$$\frac{\partial f}{\partial t} + \vec{e} \cdot \nabla f = \frac{1}{\tau} (f - f_{eq}),$$
$\tau = 3 \cdot \nu + 0.5$ is the relaxation rate towards equilibrium, where $\nu$ is kinematic viscosity. This equation replaces Navier-Stokes in CFD simulation. For our simulation we need to discretize this formula.

$$f_i(x_i + e_i \delta t, t + \delta t) = f_i(x_i, t) - \frac{\delta t}{\tau} (f_i - f_{eq}).$$

Now we have to decide how to discretize our distribution function. There are couple of possibilities for 2D case (Figure 1), I chose D2Q9.

![Figure 1: Two dimensional lattice models. From left to right: D2Q4, D2Q5, D2Q7, D2Q9.][2]
Last step is to find the formula for equilibrium distribution. It is basically an expansion of Maxwell distribution for low Mach numbers.

\[ f_{eq}^i = \rho \omega_i \left( 1 + 3(e_i \cdot \vec{v}) - \frac{3}{2} (\vec{v} \cdot \vec{v}) + \frac{9}{2} (e_i \cdot \vec{v})^2 \right) \]

where weights \( \omega_i \) are \( \frac{4}{9} \) for \( i = 0 \) (rest particle), \( \frac{1}{9} \) for \( i = 1,2,3,4 \) and \( \frac{1}{36} \) for \( i = 5,6,7,8 \).

<table>
<thead>
<tr>
<th>Navier-Stokes equation</th>
<th>Lattice Boltzmann equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \mu \nabla^2 u )</td>
<td>( \frac{\partial f}{\partial t} + e \cdot \nabla f = -\frac{1}{\tau} (f - f^{eq}) )</td>
</tr>
<tr>
<td>second-order PDE</td>
<td>first-order PDE</td>
</tr>
<tr>
<td>need to treat the non-linear convective term ( u \cdot \nabla u )</td>
<td>avoids convective term, convection becomes simple advection</td>
</tr>
<tr>
<td>need to solve Poisson equation for the pressure ( p )</td>
<td>pressure ( p ) is obtained from equation of state</td>
</tr>
</tbody>
</table>

Figure 2: Comparison between Navier-Stokes equation and Lattice Boltzmann equation. [2]

1.2 Macroscopic quantities

Since we know particle distribution \( f \) in every time step obtaining quantities like density \( \rho \),

\[ \rho = \sum_{i=0}^{8} f_i \]

velocity \( \vec{v} \)

\[ \rho = \sum_{i=0}^{8} e_i \cdot f_i \]

and momentum \( \vec{m} \) is straightforward.

\[ \vec{m} = \rho \cdot \vec{v} \]

2 Model implementation

The base class in my simulation is called Node (see the code below)

```cpp
class Node{
  public:
    // for BGK and MRT model
    double f[9]; // distribution function
    double fEq[9]; // equilibrium distribution function
    Vec2 velocity;
    double density;
    bool isFluid;
    // only for MRT collision model
    double m[9]; // moments
    double mEq[9]; // equilibrium moments
};
```

To increase stability of LBM I use MRT (Multiple relaxation time, which is described here [3]) collision operator instead of standard BGK.

2.1 Boundary conditions

At the inlet I have parabolic velocity profile which is realized using Zou/He velocity boundary condition[4]. At the outlet I assume \( \frac{\partial u}{\partial n} = 0 \) (which is not the best solution for unsteady flows[5]), where \( n \) is the normal vector of the outlet. The upper and lower boundaries and circular cylinder are solid, I use mid-grid bounce-back rule on them[6].
2.2 Visualization

There are three modes for field visualization in my program:

(a) velocity field (press u on keyboard):
   - red color corresponds to velocity in the right direction,
   - blue color corresponds to velocity in the left direction,
   - green color corresponds to absolute value of velocity y component,
(b) speed field (press s),
(c) vorticity field (press v) - best for vortex street visualization.

Moreover pressing 'a' releases marked particles which are advected by the velocity field.

3 Simulation results

3.1 Flow behind a circular cylinder

Let me introduce a Reynolds number

\[ Re = \frac{u \cdot L}{\nu} \]

where \( u \) is the max velocity at the inlet, \( L \) is the characteristic length (diameter of the cylinder) and \( \nu \) is the kinematic viscosity.

To visualize vortex street we need to put particles in time-dependent velocity field. Literature[7] suggest that we should expect vortex to shed circle 60 Reynolds number.

I simulated flow for 30, 40, 50, 60, 80, 100, and 150 Reynolds number and put particles after 20 000 time step. The domain size is 402 x 83. Maximum of inlet velocity parabolic profile is 0.15.

Figure 3: Flow behind a circular cylinder. Re = 30. Advected particles created streaklines.

Figure 4: Flow behind a circular cylinder. Re = 40. Advected particles created streaklines.
Figure 4: Flow behind a circular cylinder. $Re = 50$. Advected particles created streaklines.

Figure 6: Flow behind a circular cylinder. $Re = 60$. Advected particles created streaklines.

Figure 7: Flow behind a circular cylinder. $Re = 80$. Advected particles created streaklines.

Figure 8: Flow behind a circular cylinder. $Re = 100$. Advected particles created streaklines.

Figure 8: Flow behind a circular cylinder. $Re = 150$. Advected particles created streaklines.
References


[2] Igor Mele, 

[3] Joost Geerdink, Entropic and multiple relaxation time lattice Boltzmann methods compared for time harmonic flows, MSc, University of Amsterdam, 2008


