Lattice Boltzmann Method

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Abstract

The flow of a fluid is described by Navier-Stokes equation[1]. Unfortunately, it can be solved analytically only for small group of problems. In this project I will use Multiple Relaxation Time Lattice Boltzmann method to reproduce von Karman vortex street

1 Model description

Lattice Boltzmann method evolved out of Lattice Gas Automata, which simulated a gas through particles at discrete points in space, represented by boolean variables. LBM replaces the boolean variables of LGA with the discretized probability distribution functions f which eliminates the need for ensemble averaging.

1.1 Lattice Boltzmann equation

Let us start with the continuous Boltzmann equation which describes time evolution of our hydrodynamic system,

$$\frac{\partial f}{\partial t} + \vec{e} \cdot \boldsymbol{\nabla} f = \Omega_k,$$

where f is distribution function, $\vec{e}$ is velocity and Ω_k is collision operator. In simplest case Ω_k is linear operator in BGK approximation:

$$\frac{\partial f}{\partial t} + \vec{e} \cdot \boldsymbol{\nabla} f = \frac{1}{\tau} \left(f - f^{eq} \right),$$

$\tau = 3 \cdot \nu + 0.5$ is the relaxation rate towards equilibrium, where ν is kinematic viscosity. This equation replaces Navier-Stokes in CFD simulation. For out simulation we need to discretisize this formula.

$$f_i(x_i + e_i \delta t, t + \delta t) = f_i(x_i, t) - \frac{\delta t}{\tau} \left(f_i - f_i^{eq} \right),$$

Now we have to decide how to discretisize our distribution function. There are couple of possibilities for 2D case (Figure 1), I chose D2Q9.

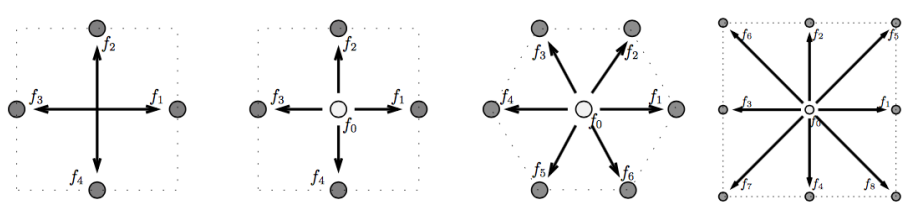

Figure 1: Two dimensional lattice models. From left to right: D2Q4, D2Q5, D2Q7, D2Q9. [2]

Last step is to find the formula for equilibrium distribution. It is basically an expansion of Maxwell distribution for low Mach numbers.

$$f_i^{eq} = \rho\omega_i \left(1 + 3(\vec{e}_i \cdot \vec{v}) - \frac{3}{2}(\vec{v} \cdot \vec{v}) + \frac{9}{2}(\vec{e}_i \cdot \vec{v})^2\right)$$

where weights $omega_i$ are $\frac{4}{9}$ for i = 0 (rest particle), $\frac{1}{9}$ for i = 1,2,3,4 and $\frac{1}{36}$ for i = 5,6,7,8.

Navier-Stokes equation	**Lattice Boltzmann equation**
$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}$	$\frac{\partial f}{\partial t} + \mathbf{e}\cdot\nabla f = -\frac{1}{\tau}\left(f - f^{\mathrm{EQ}}\right)$
second-order PDE	first-order PDE
need to treat the non-linear convective term $\mathbf{u}\cdot\nabla\mathbf{u}$	avoids convective term, convection becomes simple advection
need to solve Poisson equation for the pressure p	pressure p is obtained from equation of state

Figure 2: Comparison between Navier-Stokes equation and Lattice Boltzmann equation. [2]

1.2 Macroscopic quantities

Since we know particle distribution f in every time step obtaining quantities like density ρ,

$$\rho = \sum_{i=0}^{8} f_i$$

velocity $\vec{v}$

$$\rho = \sum_{i=0}^{8} \vec{e}_i \cdot f_i$$

and momentum $\vec{m}$ is straightforward.

$$\vec{m} = \rho \cdot \vec{v}$$

2 Model implementation

The base class in my simulation is called *Node* (see the code below)

```
class Node{
 public:
    //for BGK and MRT model
    double f[9]; //distribution function
    double fEq[9]; //equilibrium distribution function
    Vec2 velocity;
    double density;
    bool isFluid;
    //only for MRT collision model
    double m[9]; //moments
    double mEq[9]; //equilibrium moments

};
```

To increase stability of LBM I use MRT (Multiple relaxation time, which is described here [3]) collision operator instead of standard BGK.

2.1 Boundary conditions

At the inlet I have parabolic velocity profile which is realized using Zou/He velocity boundary condition[4]. At the outlet I assume $\frac{\partial v}{\partial n} = 0$ (which is not the best solution for unsteady flows[5]), where n is the normal vector of the outlet. The upper and lower boundaries and circular cylinder are solid, I use mid-grid bounce-back rule on them[6].

2.2 Visualization

There are three modes for field visualization in my program:

(a) velocity field (press u on keyboard):

- red color corresponds to velocity in the right direction,
- blue color corresponds to velocity in the left direction,
- green color corresponds to absolute value of velocity y component,

(b) speed field (press s),

(c) vorticity field (press v) - best for vortex street visualization.

Moreover pressing $'a'$ releases marked particles which are advected by the velocity field.

3 Simulation results

3.1 Flow behind a circular cylinder

Let me introduce a Reynolds number

$$Re = \frac{u \cdot L}{\nu}$$

where u is the max velocity at the inlet, L is the characteristic length (diameter of the cylinder) and ν is the kinematic viscosity.
To visualize vortex street we need to put particles in time-dependent velocity field. Literature[7] suggest that we should expect vortex to shed circle 60 Reynolds number.
I simulated flow for 30, 40, 50, 60, 80, 100, and 150 Reynolds number and put particles after 20 000 time step. The domain size is 402 x 83. Maximum of inlet velocity parabolic profile is 0.15.

Figure 3: Flow behind a circular cyllinder. Re = 30. Advected particles created streaklines.

Figure 4: Flow behind a circular cyllinder. Re = 40. Advected particles created streaklines.

Figure 4: Flow behind a circular cyllinder. Re = 50. Advected particles created streaklines.

Figure 6: Flow behind a circular cyllinder. Re = 60. Advected particles created streaklines.

Figure 7: Flow behind a circular cyllinder. Re = 80. Advected particles created streaklines.

Figure 8: Flow behind a circular cyllinder. Re = 100. Advected particles created streaklines.

Figure 8: Flow behind a circular cyllinder. Re = 150. Advected particles created streaklines.

References

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