Quantum fields on curved momentum space

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  Free falling vs. fiducial observer in Schwarzschild background: *Hawking effect*

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Less known context in which *free* QFT manifests non-trivial features:
field quantization on *group manifold* **momentum space**

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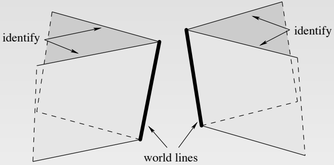

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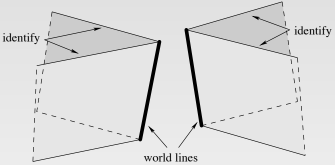

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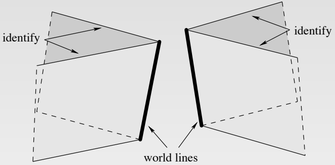

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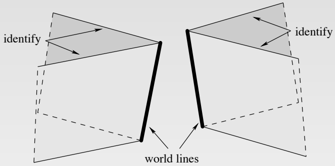

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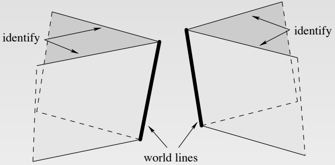

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Momenta become coordinate functions on a non-abelian group!

Outline

- **Warm up: free field quantization and observables**
- **"Bending" phase space in 3d: running spectral dimension**
- κ**-quantum fields: two-point function and a new quantization ambiguity**

Field quantization

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Classical fields	Quantum fields
state $=$ *point in phase space* $\phi \in \mathcal{S}$	**state** $=$ *ray in complex Hilbert space* $\mathcal{H}$
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- The coproduct Δ extends the action of elements of $\mathcal{P}$ to multiparticle states

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define *group Fourier transform* (Freidel and Majid 2005)

$$\mathcal{F}(f)(x) = \int d\mu_H(g) f(g)\, e_g(x)\,,$$

maps fields *on the group manifold* to fields on a *dual spacetime*...

Group-valued plane waves and deformed symmetries

...the group structure induces a non-commutative $\star$-**product** for plane waves

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non-abelian composition of momenta = **non-trivial coproduct**

$$\boxed{\Delta P_a = P_a \otimes \mathbf{1} + \mathbf{1} \otimes P_a + G\,\epsilon_{abc} P_b \otimes P_c + \mathcal{O}(G^2)}$$

the *smoking gun* of symmetry deformation...P_a belong to a non-trivial Hopf algebra with G as a deformation parameter!

An application: heath kernel and spectral dimension

Fractal space in semiclassical gravity

MA and E. Alesci: in progress...still details to be fixed

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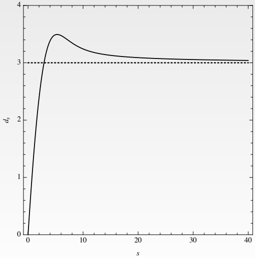
and calculate the *spectral dimension* $d_s = -2\frac{\partial \log \tilde{T}rK}{\partial \log s}$... (plot for $G=1$, $m=0$)



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    $$-\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = \kappa^2 \, ; \qquad \eta_0 + \eta_4 > 0$$

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  - dual Lie algebra “space-time” coordinates

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- consider a one-parameter group splitting of B, $0 \leq |\beta| \leq 1$

  $$e_p \equiv e^{-i\frac{1-\beta}{2}p^0 x_0} e^{ip^j x_j} e^{-i\frac{1+\beta}{2}p^0 x_0} \, .$$

  with momentum composition rules and "antipodes"

  $$p \oplus_\beta q = (p^0 + q^0 ; \, p^j \; e^{\frac{1-\beta}{2\kappa}q^0} + q^j \; e^{-\frac{1+\beta}{2\kappa}p^0}) \, , \qquad \ominus_\beta p = (-p^0 ; \, -e^{\frac{-\beta}{\kappa}p^0} p^i) \, .$$

  each choice of β corresponds to a *choice of coordinates* on the group manifold.

κ-Poincaré II

for $\beta = 1$ we have "flat slicing" coordinates

$$\begin{aligned}
\eta_0(p_0, \mathbf{p}) &= \kappa \sinh p_0/\kappa + \frac{\mathbf{p}^2}{2\kappa}\, e^{p_0/\kappa}, \\
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\eta_4(p_0, \mathbf{p}) &= \kappa \cosh p_0/\kappa - \frac{\mathbf{p}^2}{2\kappa}\, e^{p_0/\kappa}.
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\Delta(N_j) &= N_j \otimes 1 + e^{-P_0/\kappa} \otimes N_j + \frac{\epsilon_{jkl}}{\kappa} P_k \otimes M_l \\
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in the limit $\kappa \longrightarrow \infty$ recover ordinary Poincaré algebra

A new quantization ambiguity

Functions on the deformed mass-shell $\phi \in C^\infty(M^\kappa_m)$ defined by the "wave equation"

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No preferred choice of translation generators from which we can define an **energy** coordinate on M_m^κ and thus **no preferred choice of J and P^+** to define one-particle Hilbert space.

κ-particle Hilbert space and two-point function

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- Hilbert space $= C^\infty(M^\kappa)$ functions on deformed mass shell $\omega_\kappa^\pm(\mathbf{p}) = -\kappa \log\left(1 \mp \frac{|\mathbf{p}|}{\kappa}\right)$ equipped with inner product

$$(\phi_1, \phi_2)_\kappa = \int_{M_m^{\kappa+}} \frac{d\tilde{\mu}(\mathbf{p})}{2|\mathbf{p}|}\ \bar{\phi}_1(\mathbf{p})\, \phi_2(\mathbf{p})$$

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group Fourier transform $\Rightarrow$ *NC space-time* counterparts of functions on the mass-shell

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field mode operators (MA, Phys. Rev. **D83**, 025025 (2011)): $\hat{\phi}_\kappa(\mathbf{p}) \equiv \frac{1}{2|\mathbf{p}|}\left(a(\mathbf{p}) + \mathcal{J}_\ominus(\mathbf{p})\,a^\dagger(\ominus\mathbf{p})\right)$

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Fundamental building block of κ-QFT: the two-point function

$$G_+(\mathbf{p}_1, t; \mathbf{p}_2, s) \equiv \langle 0|\hat{\phi}_\kappa(\mathbf{p}_1, t)\hat{\phi}_\kappa(\mathbf{p}_2, s)|0\rangle = \frac{\delta^3(\mathbf{p}_1 \oplus \mathbf{p}_2)}{2|\mathbf{p}_1|} \mathcal{J}_\ominus(\mathbf{p}_1) \exp(-i\omega_\kappa(\mathbf{p}_1)(t-s))$$

work in progress (with J. Kowalski-Glikman and T. Trzesniewski) with Feynman propagator and "zoology" of Green functions...

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$$\sigma^\kappa(|\mathbf{k_1}\rangle \otimes |\mathbf{k_2}\rangle) = |(1-\epsilon_1)\,\mathbf{k_2}\rangle \otimes |(1-\epsilon_2)^{-1}\,\mathbf{k_1}\rangle\,, \qquad \epsilon_i = \frac{|\mathbf{k}_i|}{\kappa}$$

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$$\sigma^\kappa(|\mathbf{k_1}\rangle \otimes |\mathbf{k_2}\rangle) = |(1-\epsilon_1)\,\mathbf{k_2}\rangle \otimes |(1-\epsilon_2)^{-1}\,\mathbf{k_1}\rangle\,, \qquad \epsilon_i = \frac{|\mathbf{k}_i|}{\kappa}$$

- E.g. there will be **two** 2-particle states

$$\begin{aligned} |\mathbf{k_1 k_2}\rangle_\kappa &= \frac{1}{\sqrt{2}}\left[|\,\mathbf{k_1}\rangle \otimes |\,\mathbf{k_2}\rangle + |\,(1-\epsilon_1)\mathbf{k_2}\rangle \otimes |\,(1-\epsilon_2)^{-1}\mathbf{k_1}\rangle\right] \\ |\mathbf{k_2 k_1}\rangle_\kappa &= \frac{1}{\sqrt{2}}\left[|\,\mathbf{k_2}\rangle \otimes |\,\mathbf{k_1}\rangle + |\,(1-\epsilon_2)\mathbf{k_1}\rangle \otimes |\,(1-\epsilon_1)^{-1}\mathbf{k_2}\rangle\right] \end{aligned}$$

  with **same energy** and different linear momentum

$$\begin{aligned} \mathbf{K}_{12} = \mathbf{k_1} \oplus \mathbf{k_2} &= \mathbf{k_1} + (1-\epsilon_1)\mathbf{k_2} \\ \mathbf{K}_{21} = \mathbf{k_2} \oplus \mathbf{k_1} &= \mathbf{k_2} + (1-\epsilon_2)\mathbf{k_1} \end{aligned}$$

κ-Fock space

In ordinary QFT the full (bosonic) **Fock space** is obtained from symmetrized tensor prods of $\mathcal{H}$

In the κ-**deformed case** try to proceed in an analogous way BUT...
the symmetrized state

$$1/\sqrt{2}\,(|\mathbf{k_1}\rangle \otimes |\mathbf{k_2}\rangle + |\mathbf{k_2}\rangle \otimes |\mathbf{k_1}\rangle)$$

is NOT an eigenstate of P_μ due to the role of **non-trivial coproduct**

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given n-different modes one has $n!$ different n-particle states, one for each permutation of the n modes $\mathbf{k_1}\,, \mathbf{k_2} \ldots \mathbf{k_n}$

Hidden entanglement at the Planck scale

The non-trivial algebraic structure of κ-translations endows the Fock space with a "fine structure"

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- the 2-mode Hilbert space becomes $\mathcal{H}^2_\kappa \cong \mathcal{S}_2\mathcal{H}^2 \otimes \mathbb{C}^2$, where $\mathcal{S}_2\mathcal{H}^2$ is the ordinary symmetrized 2-mode Hilbert space and our states can be written as

$$\begin{aligned} |\epsilon\rangle \otimes |\uparrow\rangle &= |\mathbf{k_1}\mathbf{k_2}\rangle_\kappa \\ |\epsilon\rangle \otimes |\downarrow\rangle &= |\mathbf{k_2}\mathbf{k_1}\rangle_\kappa \end{aligned}$$

with $\epsilon = \epsilon(\mathbf{k_1}) + \epsilon(\mathbf{k_2})$

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- e.g. the state superposition of two total "classical" energies $\epsilon_A = \epsilon(\mathbf{k}_{1A}) + \epsilon(\mathbf{k}_{2A})$ and $\epsilon_B = \epsilon(\mathbf{k}_{1B}) + \epsilon(\mathbf{k}_{2B})$ can be entangled with the additional hidden modes e.g.

$$|\Psi\rangle = 1/\sqrt{2}(|\epsilon_A\rangle \otimes |\uparrow\rangle + |\epsilon_B\rangle \otimes |\downarrow\rangle)$$

...possible consequences for phenomenology?

(MA., D. Benedetti, [arXiv:0809.0889 [hep-th]]. MA., A. Marciano, [arXiv:0707.1329 [hep-th]]. MA, A. Hamma, S. Severini, [arXiv:0806.2145 [hep-th]].)

Conclusions

- Relativistic symmetries can be deformed to allow **“curvature” for momentum space**
- Strong motivations to look at such deformations from **2+1 gravity coupled to relativistic particles**...application: appearance of *running spectral dimension*
- Quantization of (free) field theories with group valued momenta leads to **ambiguities** related to the different choices of translation generators...physical interpretation of such ambiguities?
- What role of **deformed 2-point functions** for “trans-planckian” issues in semiclassical gravity (BH evaporation, Inflation)??
- At the multiparticle level the non-trivial behaviour of field modes leads to a **fine structure of Fock space**: interesting **entanglement** phenomena can take place

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Buon compleanno Prof. Lukierski!