

SPACE-TIME EMERGING FROM QUARK ALGEBRA

Richard Kerner

*Université Pierre et Marie Curie,
Paris, France*

XXIX Max Born Symposium
Wroclaw, June 2011

Introduction

- ▶ The Pauli exclusion principle, according to which two electrons cannot be in the same state characterized by identical quantum numbers, is one of the most important foundations of quantum physics.

Introduction

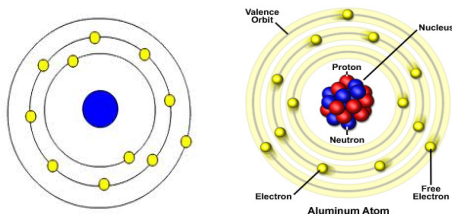
- ▶ The Pauli exclusion principle, according to which two electrons cannot be in the same state characterized by identical quantum numbers, is one of the most important foundations of quantum physics.
- ▶ Not only it does explain the structure of atoms and the periodic table of elements, but it also guarantees the stability of matter preventing its collapse.

Introduction

- ▶ The Pauli exclusion principle, according to which two electrons cannot be in the same state characterized by identical quantum numbers, is one of the most important foundations of quantum physics.
- ▶ Not only it does explain the structure of atoms and the periodic table of elements, but it also guarantees the stability of matter preventing its collapse.
- ▶ The link between the exclusion principle and particle's spin, known as the “spin-and-statistic theorem”, is one of the deepest results in quantum field theory.

Introduction

Because fermionic operators must satisfy anti-commutation relations $\psi^a \psi^b = -\psi^b \psi^a$, two electrons (or other fermions) cannot coexist in the same state.



For the principal quantum number n there are only $2 \times n^2$ electrons in different states.

Introduction

- ▶ In purely algebraical terms Pauli's exclusion principle amounts to the anti-symmetry of wave functions describing two coexisting particle states.

Introduction

- ▶ In purely algebraical terms Pauli's exclusion principle amounts to the anti-symmetry of wave functions describing two coexisting particle states.
- ▶ The easiest way to see how the principle works is to apply Dirac's formalism in which wave functions of particles in given state are obtained as products between the “bra” and “ket” vectors.

Introduction

Consider the wave function of a particle in the state $|x\rangle$,

$$\Phi(x) = \langle \psi | x \rangle. \quad (1)$$

A two-particle state of $(|x\rangle, |y\rangle)$ is a tensor product

$$|\psi\rangle = \sum \Phi(x, y) (|x\rangle \otimes |y\rangle). \quad (2)$$

If the wave function $\Phi(x, y)$ is anti-symmetric, i.e. if it satisfies

$$\Phi(x, y) = -\Phi(y, x), \quad (3)$$

then $\Phi(x, x) = 0$ and such states have vanishing probability.

Introduction

Conversely, let us suppose that $\Phi(x, x)$ vanish. This remains valid in any basis provided the new basis $|x' \rangle, |y' \rangle$ was obtained from the former one via an unitary transformation. Let us form an arbitrary state being a linear combination of $|x \rangle$ and $|y \rangle$,

$$|z \rangle = \alpha |x \rangle + \beta |y \rangle, \quad \alpha, \beta \in \mathbf{C},$$

and let us form the wave function of a tensor product of such a state with itself:

$$\Phi(z, z) = \langle \psi | (\alpha |x \rangle + \beta |y \rangle) \otimes (\alpha |x \rangle + \beta |y \rangle), \quad (4)$$

Introduction

- which develops as follows:

$$\begin{aligned}
 & \alpha^2 \langle \psi | (x, x) \rangle + \alpha\beta \langle \psi | (x, y) \rangle \\
 & + \beta\alpha \langle \psi | (y, x) \rangle + \beta^2 \langle \psi | (y, y) \rangle = \\
 & = \Phi(x, y) = \alpha^2 \Phi(x, x) + \alpha\beta \Phi(x, y) + \beta\alpha \Phi(y, x) + \beta^2 \Phi(y, y).
 \end{aligned}
 \tag{5}$$

Introduction

- ▶ which develops as follows:

$$\begin{aligned}
 & \alpha^2 \langle \psi | (x, x) \rangle + \alpha\beta \langle \psi | (x, y) \rangle \\
 & + \beta\alpha \langle \psi | (y, x) \rangle + \beta^2 \langle \psi | (y, y) \rangle = \\
 & = \Phi(x, y) = \alpha^2 \Phi(x, x) + \alpha\beta \Phi(x, y) + \beta\alpha \Phi(y, x) + \beta^2 \Phi(y, y).
 \end{aligned}
 \tag{5}$$

- ▶ Now, as $\Phi(x, x) = 0$ and $\Phi(y, y) = 0$, the sum of remaining two terms will vanish if and only if (3) is satisfied, i.e. if $\Phi(x, y)$ is anti-symmetric in its two arguments.

Introduction

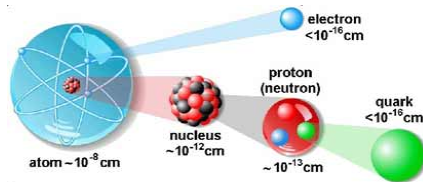
After second quantization, when the states are obtained with creation and annihilation operators acting on the vacuum, the anti-symmetry is encoded in the anti-commutation relations

$$\psi(x)\psi(y) + \psi(y)\psi(x) = 0 \quad (6)$$

where $\psi(x) | 0 \rangle = | \psi \rangle$.

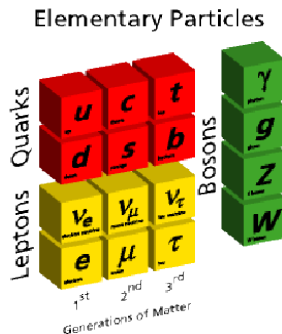
Introduction

According to present knowledge, the ultimate undivisible and indestructible constituents of matter, called **atoms** by ancient Greeks, are in fact the **QUARKS**, carrying fractional electric charges and baryonic numbers, two features that appear to be **indestructible and conserved under any circumstances**.



Introduction: Quarks and Leptons

The carriers of elementary charges also go by packs of three: three families of quarks, and three types of leptons.



Three Generations of Matter (Fermions)

	I	II	III	
mass	2.4 MeV	1.27 GeV	171.2 GeV	0
charge	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0
spin	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
name	u up	c charm	t top	γ photon
Quarks	4.8 MeV $-\frac{1}{3}$ $\frac{1}{2}$ d down	104 MeV $-\frac{1}{3}$ $\frac{1}{2}$ s strange	4.2 GeV $-\frac{1}{3}$ $\frac{1}{2}$ b bottom	0 0 1 g gluon
Leptons	~ 2.2 eV 0 $\frac{1}{2}$ ν_e electron neutrino	~ 0.17 MeV 0 $\frac{1}{2}$ ν_μ muon neutrino	~ 15.5 MeV 0 $\frac{1}{2}$ ν_τ tau neutrino	51.2 GeV 0 1 Z weak force
	0.511 MeV -1 $\frac{1}{2}$ e electron	105.7 MeV -1 $\frac{1}{2}$ μ muon	1.777 GeV -1 $\frac{1}{2}$ τ tau	80.4 GeV ± 1 1 W weak force
				Bosons (Forces)

Introduction

- In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$.

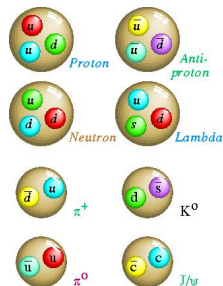
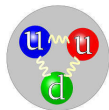
Introduction

- ▶ In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$.
- ▶ Only *three* quarks or anti-quarks can coexist inside a fermionic baryon (respectively, anti-baryon), and a pair quark-antiquark can form a meson with integer spin.

Introduction

- ▶ In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$.
- ▶ Only *three* quarks or anti-quarks can coexist inside a fermionic baryon (respectively, anti-baryon), and a pair quark-antiquark can form a meson with integer spin.
- ▶ Besides, they must belong to different *colors*, also a three-valued set. There are two quarks in the first generation, u and d (“up” and “down”), which may be considered as two states of a more general object, just like proton and neutron in $SU(2)$ symmetry are two isospin components of a nucleon doublet.

Introduction



Baryons (hadrons) are composed of quarks,
which cannot be observed in a free (unbound) state.

Introduction

This suggests that a convenient generalization of Pauli's exclusion principle would be that **no *three* quarks in the same state can be present in a nucleon.**

Let us require then the vanishing of wave functions representing the tensor product of *three* (but not necessarily two) identical states. That is, we require that $\Phi(x, x, x) = 0$ for any state $|x\rangle$. **As in the former case, consider an arbitrary superposition of three different states, $|x\rangle$, $|y\rangle$ and $|z\rangle$,**

$$|w\rangle = \alpha |x\rangle + \beta |x\rangle + \gamma |z\rangle$$

and apply the same criterion, $\Phi(w, w, w) = 0$.

Introduction

We get then, after developing the tensor products,

$$\begin{aligned} \Phi(w, w, w) &= \alpha^3 \Phi(x, x, x) + \beta^3 \Phi(y, y, y) + \gamma^3 \Phi(z, z, z) \\ &+ \alpha^2 \beta [\Phi(x, x, y) + \Phi(x, y, x) + \Phi(y, x, x)] + \gamma \alpha^2 [\Phi(x, x, z) + \Phi(x, z, x) + \Phi(z, x, x)] \\ &+ \alpha \beta^2 [\Phi(y, y, x) + \Phi(y, x, y) + \Phi(x, y, y)] + \beta^2 \gamma [\Phi(y, y, z) + \Phi(y, z, y) + \Phi(z, y, y)] \\ &+ \beta \gamma^2 [\Phi(y, z, z) + \Phi(z, z, y) + \Phi(z, y, z)] + \gamma^2 \alpha [\Phi(z, z, x) + \Phi(z, x, z) + \Phi(x, z, z)] \\ &+ \alpha \beta \gamma [\Phi(x, y, z) + \Phi(y, z, x) + \Phi(z, x, y) + \Phi(z, y, x) + \Phi(y, x, z) + \Phi(x, z, y)] = 0. \end{aligned}$$

The terms $\Phi(x, x, x)$, $\Phi(y, y, y)$ and $\Phi(z, z, z)$ vanish by virtue of the original assumption; in what remains, combinations preceded by various powers of independent numerical coefficients α, β and γ , must vanish separately. This is achieved if the following Z_3 symmetry is imposed on our wave functions:

$$\Phi(x, y, z) = j \Phi(y, z, x) = j^2 \Phi(z, x, y),$$

with $j = e^{\frac{2\pi i}{3}}$, $j^3 = 1$, $j + j^2 + 1 = 0$.

Note that the complex conjugates of functions $\Phi(x, y, z)$ transform under cyclic permutations of their arguments with $j^2 = \bar{j}$ replacing j in the above formula

$$\Psi(x, y, z) = j^2 \Psi(y, z, x) = j \Psi(z, x, y).$$

- Inside a hadron, not two, but three quarks in different states (colors) can coexist.
After second quantization, when the fields become operator-valued, an alternative CUBIC commutation relations seems to be more appropriate:

- Inside a hadron, not two, but three quarks in different states (colors) can coexist.

After second quantization, when the fields become operator-valued, an alternative CUBIC commutation relations seems to be more appropriate:

- Instead of

$$\psi^a \psi^b = (-1) \psi^b \psi^a$$

we can introduce

$$\theta^A \theta^B \theta^C = j \theta^B \theta^C \theta^A,$$

with $j = e^{\frac{2\pi i}{3}}$

Algebraic properties of quark states

- Our aim now is to derive the space-time symmetries from minimal assumptions concerning the properties of the most elementary constituents of matter, and the best candidates for these are quarks.

Algebraic properties of quark states

- ▶ Our aim now is to derive the space-time symmetries from minimal assumptions concerning the properties of the most elementary constituents of matter, and the best candidates for these are quarks.
- ▶ To do so, we should explore algebraic structures that would privilege **cubic** or **ternary** relations, in other words, find appropriate **cubic** or **ternary** algebras reflecting the most important properties of quark states.

The minimal requirements for the definition of quarks at the initial stage of model building are the following:

- ▶ i) The mathematical entities representing the quarks form a linear space over complex numbers, so that we could form their linear combinations with complex coefficients.

The minimal requirements for the definition of quarks at the initial stage of model building are the following:

- ▶ *i*) The mathematical entities representing the quarks form a linear space over complex numbers, so that we could form their linear combinations with complex coefficients.
- ▶ *ii*) They should also form an associative algebra, so that we could form their multilinear combinations;

The minimal requirements for the definition of quarks at the initial stage of model building are the following:

- ▶ *i*) The mathematical entities representing the quarks form a linear space over complex numbers, so that we could form their linear combinations with complex coefficients.
- ▶ *ii*) They should also form an associative algebra, so that we could form their multilinear combinations;
- ▶ *iii*) There should exist two isomorphic algebras of this type corresponding to quarks and anti-quarks, and the conjugation that maps one of these algebras onto another, $\mathcal{A} \rightarrow \bar{\mathcal{A}}$.

The minimal requirements for the definition of quarks at the initial stage of model building are the following:

- ▶ *i*) The mathematical entities representing the quarks form a linear space over complex numbers, so that we could form their linear combinations with complex coefficients.
- ▶ *ii*) They should also form an associative algebra, so that we could form their multilinear combinations;
- ▶ *iii*) There should exist two isomorphic algebras of this type corresponding to quarks and anti-quarks, and the conjugation that maps one of these algebras onto another, $\mathcal{A} \rightarrow \bar{\mathcal{A}}$.
- ▶ *iv*) The three quark (or three anti-quark) and the quark-anti-quark combinations should be distinguished in a certain way, for example, they should form a subalgebra in the enveloping algebra spanned by the generators.

The simplest discrete groups

- The groups of permutations, or symmetric groups, denoted by S_n , consist of all permutation operations acting on any set containing n items. The dimension of an S_n group is therefore equal to $n!$. Cyclic permutations of n elements form an n -dimensional subgroup of S_n denoted by Z_n .

The simplest discrete groups

- ▶ The groups of permutations, or symmetric groups, denoted by S_n , consist of all permutation operations acting on any set containing n items. The dimension of an S_n group is therefore equal to $n!$. Cyclic permutations of n elements form an n -dimensional subgroup of S_n denoted by Z_n .
- ▶ The S_2 group contains only two elements, the identity keeping two items unchanged, and the only non-trivial permutation of two items, $(ab) \rightarrow (ba)$. This permutation is cyclic, so the S_2 group coincides with its Z_2 subgroup.

The simplest discrete groups

The simplest representations of the Z_2 group are realised via its actions on the complex numbers, \mathbf{C}^1 . Three different inversions can be introduced, each of them generating a different representation of Z_2 in the complex plane \mathbf{C}^1 :

- i) the sign inversion, $z \rightarrow -z$;
- ii) complex conjugation, $z \rightarrow \bar{z}$;
- iii) the combination of both, $z \rightarrow -\bar{z}$.

One should not forget about the fourth possibility, the trivial representation attributing the identity transformation to the two elements of the group, including the non-trivial one:

- iv) the identity transformation, $z \rightarrow z$.

The principle of covariance

Any meaningful quantity described by a set of functions $\psi^A(x^\mu)$, $A, B, \dots = 1, 2, \dots, N$, $\mu, \nu, \dots = 0, 1, 2, 3$ defined on the Minkowskian space-time must be a representation of the Lorentz group, i.e. it should transform following one of its representations:

$$\psi^{A'}(x^{\mu'}) = \psi^{A'}(\Lambda_{\rho}^{\mu'} x^{\rho}) = S_B^{A'}(\Lambda_{\rho}^{\mu'}) \psi^B(x^{\rho}). \quad (7)$$

which can be written even more concisely,

$$\psi(x') = S(\Lambda)(\psi(x)). \quad (8)$$

The important assumption here being the representation property of the linear transformations $S(\Lambda)$:

$$S(\Lambda_1)S(\Lambda_2) = S(\Lambda_1\Lambda_2). \quad (9)$$

Covariance principle: the discrete case

A similar principle can be formulated in the discrete case of permutation groups, in particular for the Z_2 group. Instead of a set of functions defined on the space-time, we consider the mapping of two indices into the complex numbers, i.e. a matrix or a two-valenced complex-valued tensor. Under the non-trivial permutation π of indices its value should change according to one of the possible representations of Z_2 in the complex plane. This leads to the following four possibilities:

Covariance principle: the discrete case

i) The trivial representation defines the symmetric tensors:

$$S_{\pi(AB)} = S_{BA} = S_{AB},$$

ii) The sign inversion defines the anti-symmetric tensors:

$$A_{\pi(CD)} = A_{DC} = -A_{CD},$$

iii) The complex conjugation defines the hermitian tensors:

$$H_{\pi(AB)} = H_{BA} = \bar{H}_{AB},$$

iv) $(-1) \times$ complex conjugation defines the anti-hermitian tensors.

$$T_{\pi(AB)} = T_{BA} = -\bar{T}_{AB},$$

The symmetric S_3 group

- ▶ The symmetric group S_3 containing all permutations of three different elements is a special case among all symmetry groups S_N . It is exceptional because it is the first in the row to be non-abelian, and the last one that possesses a faithful representation in the complex plane \mathbb{C}^1 .

The symmetric S_3 group

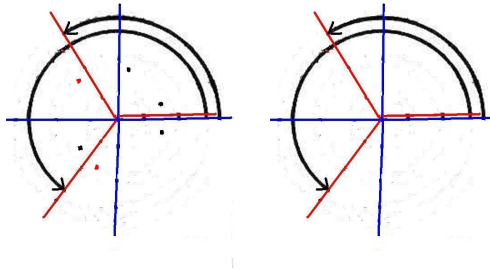
- ▶ The symmetric group S_3 containing all permutations of three different elements is a special case among all symmetry groups S_N . It is exceptional because it is the first in the row to be non-abelian, and the last one that possesses a faithful representation in the complex plane \mathbb{C}^1 .
- ▶ It contains six elements, and can be generated with only two elements, corresponding to one cyclic and one odd permutation, e.g. $(abc) \rightarrow (bca)$, and $(abc) \rightarrow (cba)$. All permutations can be represented as different operations on complex numbers as follows.

The cyclic group Z_3

Let us denote the primitive third root of unity by $j = e^{2\pi i/3}$.

The cyclic abelian subgroup Z_3 contains three elements corresponding to the three cyclic permutations, which can be represented via multiplication by j , j^2 and $j^3 = 1$ (the identity).

$$\begin{pmatrix} ABC \\ ABC \end{pmatrix} \rightarrow \mathbf{1}, \quad \begin{pmatrix} ABC \\ BCA \end{pmatrix} \rightarrow \mathbf{j}, \quad \begin{pmatrix} ABC \\ CAB \end{pmatrix} \rightarrow \mathbf{j}^2, \quad (10)$$



The six S_3 symmetry transformations contain the identity, two rotations, one by 120° , another one by 240° , and three reflections, in the x -axis, in the j -axis and in the j^2 -axis. The Z_3 subgroup contains only the three rotations.

Representation of S_3 in the complex plane

Odd permutations must be represented by idempotents, i.e. by operations whose square is the identity operation. We can make the following choice:

$$\begin{pmatrix} ABC \\ CBA \end{pmatrix} \rightarrow (z \rightarrow \bar{z}), \quad \begin{pmatrix} ABC \\ BAC \end{pmatrix} \rightarrow (z \rightarrow \hat{z}), \quad \begin{pmatrix} ABC \\ CBA \end{pmatrix} \rightarrow (z \rightarrow z^*), \quad (11)$$

Here the bar $(z \rightarrow \bar{z})$ denotes the complex conjugation, i.e. the reflection in the real line, the hat $z \rightarrow \hat{z}$ denotes the reflection in the root j^2 , and the star $z \rightarrow z^*$ the reflection in the root j . The six operations close in a non-abelian group with six elements, and the corresponding multiplication table is shown in the following table:

The group S_3 - the multiplication table

	1	j	j^2	—	\wedge	*
1	1	j	j^2	—	\wedge	*
j	j	j^2	1	*	—	\wedge
j^2	j^2	1	j	\wedge	*	—
—	—	\wedge	*	1	j	j^2
\wedge	\wedge	*	—	j^2	1	j
*	*	—	\wedge	j	j^2	1

Table I: The multiplication table for the S_3 symmetric group

Basic definitions and properties

Let us introduce N generators spanning a linear space over complex numbers, satisfying the following cubic relations:

$$\theta^A \theta^B \theta^C = j \theta^B \theta^C \theta^A = j^2 \theta^C \theta^A \theta^B, \quad (12)$$

with $j = e^{2i\pi/3}$,

the primitive root of 1. We have $1 + j + j^2 = 0$ and $\bar{j} = j^2$.

Basic definitions and properties

We shall also introduce a similar set of *conjugate* generators, $\bar{\theta}^{\dot{A}}, \dot{A}, \dot{B}, \dots = 1, 2, \dots, N$, satisfying similar condition with j^2 replacing j :

$$\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2 \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} = j \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}, \quad (13)$$

Let us denote this algebra by \mathcal{A} .

The Z_3 graded algebra \mathcal{A}

- Let us denote the algebra spanned by the θ^A generators by \mathcal{A} . We shall endow it with a natural Z_3 grading, considering the generators θ^A as grade 1 elements, and their conjugates $\bar{\theta}^{\dot{A}}$ being of grade 2.

The Z_3 graded algebra \mathcal{A}

- ▶ Let us denote the algebra spanned by the θ^A generators by \mathcal{A} . We shall endow it with a natural Z_3 grading, considering the generators θ^A as grade 1 elements, and their conjugates $\bar{\theta}^{\dot{A}}$ being of grade 2.
- ▶ The grades add up modulo 3, so that the products $\theta^A \theta^B$ span a linear subspace of grade 2, and the cubic products $\theta^A \theta^B \theta^C$ being of grade 0.

The Z_3 graded algebra \mathcal{A}

- ▶ Let us denote the algebra spanned by the θ^A generators by \mathcal{A} . We shall endow it with a natural Z_3 grading, considering the generators θ^A as grade 1 elements, and their conjugates $\bar{\theta}^{\dot{A}}$ being of grade 2.
- ▶ The grades add up modulo 3, so that the products $\theta^A \theta^B$ span a linear subspace of grade 2, and the cubic products $\theta^A \theta^B \theta^C$ being of grade 0.
- ▶ Similarly, all quadratic expressions in conjugate generators, $\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}$ are of grade $2 + 2 = 4_{mod\ 3} = 1$, whereas their cubic products are again of grade 0, like the cubic products of θ^A 's.

The Z_3 graded algebra \mathcal{A}

Combined with the associativity, these cubic relations impose finite dimension on the algebra generated by the Z_3 graded generators. As a matter of fact, cubic expressions are the highest order that does not vanish identically. The proof is immediate:

$$\begin{aligned}\theta^A \theta^B \theta^C \theta^D &= j \theta^B \theta^C \theta^A \theta^D = j^2 \theta^B \theta^A \theta^D \theta^C = \\ &= j^3 \theta^A \theta^D \theta^B \theta^C = j^4 \theta^A \theta^B \theta^C \theta^D,\end{aligned}\tag{14}$$

and because $j^4 = j \neq 1$, the only solution is

$$\theta^A \theta^B \theta^C \theta^D = 0.\tag{15}$$

The Z_3 graded algebra \mathcal{A}

Therefore the total dimension of the algebra defined via the cubic relations (12) is equal to $N + N^2 + (N^3 - N)/3$: the N generators of grade 1, the N^2 independent products of two generators, and $(N^3 - N)/3$ independent cubic expressions, because the cube of any generator must be zero by virtue of (12), and the remaining $N^3 - N$ ternary products are divided by 3, also by virtue of the constitutive relations (12).

The conjugate generators $\bar{\theta}^{\dot{B}}$ span an algebra $\bar{\mathcal{A}}$ isomorphic with \mathcal{A} .

The Z_3 graded algebra \mathcal{A}

Both algebras split quite naturally into sums of linear subspaces with definite grades:

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2, \quad \bar{\mathcal{A}} = \bar{\mathcal{A}}_0 \oplus \bar{\mathcal{A}}_1 \oplus \bar{\mathcal{A}}_2,$$

The subspaces \mathcal{A}_0 and $\bar{\mathcal{A}}_0$ form zero-graded subalgebras. These algebras can be made *unital* if we add to each of them the unit element **1** acting as identity and considered as being of grade 0.

The Z_3 graded algebra \mathcal{A}

- If we want the products between the generators θ^A and the conjugate ones $\bar{\theta}^{\bar{B}}$ to be included into the greater algebra spanned by both types of generators, we should consider all possible products, which will be included in the linear subspaces with a definite grade. of the resulting algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$.

The Z_3 graded algebra \mathcal{A}

- If we want the products between the generators θ^A and the conjugate ones $\bar{\theta}^{\dot{B}}$ to be included into the greater algebra spanned by both types of generators, we should consider all possible products, which will be included in the linear subspaces with a definite grade. of the resulting algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$.
- The grade 1 component will contain now, besides the generators of the algebra \mathcal{A} , also the products like

$$\bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{D}}, \quad \theta^A \theta^B \theta^C \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{G}}, \quad \text{and} \quad \theta^A \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{G}},$$

and of course all possible monomials resulting from the permutations of factors in the above expressions.

The Z_3 graded algebra \mathcal{A}

The grade two component will contain, along with the conjugate generators $\bar{\theta}^{\dot{B}}$ and the products of two grade 1 generators $\theta^A \theta^B$, the products of the type

$$\theta^A \theta^B \theta^C \bar{\theta}^{\dot{D}} \quad \text{and} \quad \theta^A \theta^B \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}} \bar{\theta}^{\dot{G}}$$

and all similar monomials obtained via permutations of factors in the above.

The Z_3 graded algebra \mathcal{A}

Finally, the grade 0 component will contain now the binary products

$$\theta^A \bar{\theta}^{\dot{B}}, \quad \bar{\theta}^{\dot{B}} \theta^A,$$

the cubic monomials

$$\theta^A \theta^B \theta^C, \quad \bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}},$$

and the products of four and six generators, with the equal number of θ^A and $\bar{\theta}^{\dot{B}}$ generators:

$$\theta^A \theta^B \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{D}}, \quad \theta^A \bar{\theta}^{\dot{C}} \theta^B \bar{\theta}^{\dot{D}}, \quad \theta^A \theta^B \theta^C \bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}}, \quad \bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}} \theta^A \theta^B \theta^C,$$

and other expressions of this type that can be obtained by permutations of factors.

The Z_3 graded algebra \mathcal{A}

- The fact that the conjugate generators are endowed with grade 2 could suggest that they behave just like the products of two ordinary generators $\theta^A \theta^B$. However, such a choice does not enable one to make a clear distinction between the conjugate generators and the products of two ordinary ones, and it would be much better, to be able to make the difference.

The Z_3 graded algebra \mathcal{A}

- ▶ The fact that the conjugate generators are endowed with grade 2 could suggest that they behave just like the products of two ordinary generators $\theta^A \theta^B$. However, such a choice does not enable one to make a clear distinction between the conjugate generators and the products of two ordinary ones, and it would be much better, to be able to make the difference.
- ▶ Due to the binary nature of the products, another choice is possible, namely, to require the following commutation relations:

$$\theta^A \bar{\theta}^{\dot{B}} = -j \bar{\theta}^{\dot{B}} \theta^A, \quad \bar{\theta}^{\dot{B}} \theta^A = -j^2 \theta^A \bar{\theta}^{\dot{B}}, \quad (16)$$

Symmetries and tensors on Z_3 -graded algebras

- As all bilinear maps of vector spaces into numbers can be divided into irreducible symmetry classes according to the representations of the Z_2 group, so can the tri-linear mappings be distinguished by their symmetry properties with respect to the permutations belonging to the S_3 symmetry group.

Symmetries and tensors on Z_3 -graded algebras

- ▶ As all bilinear maps of vector spaces into numbers can be divided into irreducible symmetry classes according to the representations of the Z_2 group, so can the tri-linear mappings be distinguished by their symmetry properties with respect to the permutations belonging to the S_3 symmetry group.
- ▶ There are several different representations of the action of the S_3 permutation group on tensors with three indices. Consequently, such tensors can be divided into irreducible subspaces which are conserved under the action of S_3 .

- There are three possibilities of an action of Z_3 being represented by multiplication by a complex number: the trivial one (multiplication by 1), and the two other representations, the multiplication by $j = e^{2\pi i/3}$ or by its complex conjugate, $j^2 = \bar{j} = e^{4\pi i/3}$.

- There are three possibilities of an action of Z_3 being represented by multiplication by a complex number: the trivial one (multiplication by 1), and the two other representations, the multiplication by $j = e^{2\pi i/3}$ or by its complex conjugate, $j^2 = \bar{j} = e^{4\pi i/3}$.



$$T \in \mathcal{T} : \quad T_{ABC} = T_{BCA} = T_{CAB}, \quad (17)$$

(totally symmetric)

$$S \in \mathcal{S} : \quad S_{ABC} = j S_{BCA} = j^2 S_{CAB}, \quad (18)$$

(j -skew-symmetric)

$$\bar{S} \in \bar{\mathcal{S}}; \quad \bar{S}_{ABC} = j^2 \bar{S}_{BCA} = j \bar{S}_{CAB}, \quad (19)$$

(j^2 -skew-symmetric).

Tri-linear forms

- ▶ The space of all tri-linear forms is the sum of three irreducible subspaces,

$$\Theta_3 = \mathcal{T} \oplus \mathcal{S} \oplus \bar{\mathcal{S}}$$

the corresponding dimensions being, respectively,
 $(N^3 + 2N)/3$ for \mathcal{T} and $(N^3 - N)/3$ for \mathcal{S} and for $\bar{\mathcal{S}}$.

Tri-linear forms

- ▶ The space of all tri-linear forms is the sum of three irreducible subspaces,

$$\Theta_3 = \mathcal{T} \oplus \mathcal{S} \oplus \bar{\mathcal{S}}$$

the corresponding dimensions being, respectively, $(N^3 + 2N)/3$ for \mathcal{T} and $(N^3 - N)/3$ for \mathcal{S} and for $\bar{\mathcal{S}}$.

- ▶ Any three-form W_{ABC}^α mapping $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ into a vector space \mathcal{X} of dimension k , $\alpha, \beta = 1, 2, \dots, k$, so that $X^\alpha = W_{ABC}^\alpha \theta^A \theta^B \theta^C$ can be represented as a linear combination of forms with specific symmetry properties,

$$W_{ABC}^\alpha = T_{ABC}^\alpha + S_{ABC}^\alpha + \bar{S}_{ABC}^\alpha,$$

Irreducible three-linear forms

$$T_{ABC}^{\alpha} := \frac{1}{3} (W_{ABC}^{\alpha} + W_{BCA}^{\alpha} + W_{CAB}^{\alpha}), \quad (20)$$

$$S_{ABC}^{\alpha} := \frac{1}{3} (W_{ABC}^{\alpha} + j W_{BCA}^{\alpha} + j^2 W_{CAB}^{\alpha}), \quad (21)$$

$$\bar{S}_{ABC}^{\alpha} := \frac{1}{3} (W_{ABC}^{\alpha} + j^2 W_{BCA}^{\alpha} + j W_{CAB}^{\alpha}), \quad (22)$$

As in the Z_2 case, the three symmetries above define irreducible and mutually orthogonal 3-forms

The simplest case: two generators

Let us consider the simplest case of cubic algebra with two generators, $A, B, \dots = 1, 2$. Its grade 1 component contains just these two elements, θ^1 and θ^2 ; its grade 2 component contains four independent products,

$$\theta^1\theta^1, \theta^1\theta^2, \theta^2\theta^1, \text{ and } \theta^2\theta^2.$$

Finally, its grade 0 component (which is a subalgebra) contains the unit element 1 and the two linearly independent cubic products,

$$\theta^1\theta^2\theta^1 = j \theta^2\theta^1\theta^1 = j^2 \theta^1\theta^1\theta^2,$$

and

$$\theta^2\theta^1\theta^2 = j \theta^1\theta^2\theta^2 = j^2 \theta^2\theta^2\theta^1.$$

General definition of invariant forms

Let us consider multilinear forms defined on the algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$. Because only cubic relations are imposed on products in \mathcal{A} and in $\bar{\mathcal{A}}$, and the binary relations on the products of ordinary and conjugate elements, we shall fix our attention on tri-linear and bi-linear forms, conceived as mappings of $\mathcal{A} \otimes \bar{\mathcal{A}}$ into certain linear spaces over complex numbers.

Let us consider a tri-linear form ρ_{ABC}^α . We shall call this form Z_3 -invariant if we can write:

$$\begin{aligned} \rho_{ABC}^\alpha \theta^A \theta^B \theta^C &= \frac{1}{3} \left[\rho_{ABC}^\alpha \theta^A \theta^B \theta^C + \rho_{BCA}^\alpha \theta^B \theta^C \theta^A + \rho_{CAB}^\alpha \theta^C \theta^A \theta^B \right] = \\ &= \frac{1}{3} \left[\rho_{ABC}^\alpha \theta^A \theta^B \theta^C + \rho_{BCA}^\alpha (j^2 \theta^A \theta^B \theta^C) + \rho_{CAB}^\alpha j (\theta^A \theta^B \theta^C) \right], \quad (23) \end{aligned}$$

by virtue of the commutation relations (12).

General definition of invariant forms

From this it follows that we should have

$$\rho_{ABC}^{\alpha} \theta^A \theta^B \theta^C = \frac{1}{3} \left[\rho_{ABC}^{\alpha} + j^2 \rho_{BCA}^{\alpha} + j \rho_{CAB}^{\alpha} \right] \theta^A \theta^B \theta^C, \quad (24)$$

from which we get the following properties of the ρ -cubic matrices:

$$\rho_{ABC}^{\alpha} = j^2 \rho_{BCA}^{\alpha} = j \rho_{CAB}^{\alpha}. \quad (25)$$

General definition of invariant forms

Even in this minimal and discrete case, there are covariant and contravariant indices: the lower and the upper indices display the inverse transformation property. If a given cyclic permutation is represented by a multiplication by j for the upper indices, the same permutation performed on the lower indices is represented by multiplication by the inverse, i.e. j^2 , so that they compensate each other.

Similar reasoning leads to the definition of the conjugate forms $\bar{\rho}_{\dot{C}\dot{B}\dot{A}}^{\dot{\alpha}}$ satisfying the relations similar to (25) with j replaced by its conjugate, j^2 :

$$\bar{\rho}_{\dot{A}\dot{B}\dot{C}}^{\dot{\alpha}} = j \bar{\rho}_{\dot{B}\dot{C}\dot{A}}^{\dot{\alpha}} = j^2 \bar{\rho}_{\dot{C}\dot{A}\dot{B}}^{\dot{\alpha}} \quad (26)$$

Invariant forms: the two-generator case

- In the simplest case of two generators, the j -skew-invariant forms have only two independent components:

Invariant forms: the two-generator case

- ▶ In the simplest case of two generators, the j -skew-invariant forms have only two independent components:



$$\rho_{121}^1 = j \rho_{211}^1 = j^2 \rho_{112}^1,$$

$$\rho_{212}^2 = j \rho_{122}^2 = j^2 \rho_{221}^2,$$

Invariant forms: the two-generator case

- In the simplest case of two generators, the j -skew-invariant forms have only two independent components:



$$\begin{aligned}\rho_{121}^1 &= j \rho_{211}^1 = j^2 \rho_{112}^1, \\ \rho_{212}^2 &= j \rho_{122}^2 = j^2 \rho_{221}^2,\end{aligned}$$

- and we can set

$$\begin{aligned}\rho_{121}^1 &= 1, \quad \rho_{211}^1 = j^2, \quad \rho_{112}^1 = j, \\ \rho_{212}^2 &= 1, \quad \rho_{122}^2 = j^2, \quad \rho_{221}^2 = j.\end{aligned}$$

Cubic matrices

A tensor with three covariant indices can be interpreted as a “cubic matrix”. One can introduce a ternary multiplication law for cubic matrices defined below:

$$(a * b * c)_{ikl} := \sum_{pqr} a_{piq} b_{qkr} c_{rlp} \quad (27)$$

in which any *cyclic* permutation of the matrices in the product is equivalent to the same permutation on the indices:

$$(a * b * c)_{ikl} = (b * c * a)_{kli} = (c * a * b)_{lik} \quad (28)$$

Cubic matrices

- ▶ The ternary multiplication law involving summation on certain indices should be in fact interpreted as contraction of covariant and contravariant indices, which requires the introduction of an analog of a metric tensor.

Cubic matrices

- ▶ The ternary multiplication law involving summation on certain indices should be in fact interpreted as contraction of covariant and contravariant indices, which requires the introduction of an analog of a metric tensor.
- ▶ In the case introduced above, with

$$(a * b * c)_{ikl} := \sum_{pqr} a_{piq} b_{qkr} c_{rlp} \quad (29)$$

the “metric” is just the Kronecker delta:

Cubic matrices

- ▶ The ternary multiplication law involving summation on certain indices should be in fact interpreted as contraction of covariant and contravariant indices, which requires the introduction of an analog of a metric tensor.
- ▶ In the case introduced above, with

$$(a * b * c)_{ikl} := \sum_{pqr} a_{piq} b_{qkr} c_{rlp} \quad (29)$$

the “metric” is just the Kronecker delta:



$$(a * b * c)_{ikl} := a_{nij} b_{pkr} c_{slm} \delta^{jp} \delta^{rs} \delta^{mn}. \quad (30)$$

Ternary multiplication law

Under the ternary multiplication with Kronecker delta's playing the role of the metric the matrices ρ_{ABC}^α do not form a closed ternary algebra.

However, they form such an algebra if the symplectic two-form is used instead of the Kronecker delta:

$$\{\rho^\alpha, \rho^\beta, \rho^\gamma\}_{ABC} = \rho_{DAE}^\alpha, \rho_{FBG}^\beta, \rho_{HCJ}^\gamma \epsilon^{EF} \epsilon^{GH} \epsilon^{JD}, \quad (31)$$

with $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{11} = \epsilon^{22} = 0$.

Cubic matrices

If we want to keep a particular symmetry under such ternary composition, we should introduce a new composition law that follows the particular symmetry of the given type of cubic matrices. For example, let us define:

$$\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\} := \rho^{(\alpha)} * \rho^{(\beta)} * \rho^{(\gamma)} + j \rho^{(\beta)} * \rho^{(\gamma)} * \rho^{(\alpha)} + j^2 \rho^{(\gamma)} * \rho^{(\alpha)} * \rho^{(\beta)}$$

Ternary algebra of cubic matrices

Because of the symmetry of the ternary *j*-bracket one has

$$\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\}_{ABC} = j\{\rho^{(\alpha)}, \rho^{(\beta)}, \rho^{(\gamma)}\}_{BCA},$$

so that it becomes obvious that with respect to the *j*-bracket composition law the matrices $\rho^{(\alpha)}$ form a ternary subalgebra. Indeed, we have

$$\{\rho^{(1)}, \rho^{(2)}, \rho^{(1)}\} = -\rho^{(2)}; \quad \{\rho^{(2)}, \rho^{(1)}, \rho^{(2)}\} = -\rho^{(1)}; \quad (32)$$

all other combinations being proportional to the above ones with a factor j or j^2 , whereas the *j*-brackets of three identical matrices obviously vanish.

Ternary algebra of cubic matrices

Let us find the simplest representation of this ternary algebra in terms of a *j-commutator* defined in an associative algebra of matrices $M_2(\mathbf{C})$ as follows:

$$[A, B, C] := ABC + j BCA + j^2 CAB \quad (33)$$

It is easy to see that the trace of any j-bracket of three matrices must vanish; therefore, the matrices that would represent the cubic matrices $\rho^{(\alpha)}$ must be traceless.

Ternary algebra of cubic matrices

Then it is a matter of simple exercise to show that any two of the three Pauli sigma-matrices divided by $\sqrt{2}$ provide us with a representation of the ternary j-skew algebra of the ρ -matrices; e.g.

$$\sigma^1 \sigma^2 \sigma^1 + j \sigma^2 \sigma^1 \sigma^1 + j^2 \sigma^1 \sigma^1 \sigma^2 = -2 \sigma^2,$$

$$\sigma^2 \sigma^1 \sigma^2 + j \sigma^1 \sigma^2 \sigma^2 + j^2 \sigma^2 \sigma^2 \sigma^1 = -2 \sigma^1$$

Thus, it is possible to find a representation in the associative algebra of finite matrices for the non-associative j-bracket ternary algebra. A similar representation can be found for the two cubic matrices $r\bar{h}o^{(\dot{\alpha})}$ with the j^2 -skew bracket.

The invariance group of cubic matrices

- ▶ The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.

The invariance group of cubic matrices

- ▶ The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.
- ▶ Let $U_A^{A'}$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U_B^{B'} \theta^B$.

The invariance group of cubic matrices

- ▶ The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.
- ▶ Let $U_A^{A'}$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U_B^{B'} \theta^B$.
- ▶ We are looking for the solution of the covariance condition for the ρ -matrices:

$$\Lambda_{\beta}^{\alpha'} \rho_{ABC}^{\beta} = U_A^{A'} U_B^{B'} U_C^{C'} \rho_{A'B'C'}^{\alpha'}. \quad (34)$$

The invariance group of cubic matrices

- Now, $\rho_{121}^1 = 1$, and we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ -matrix on the right-hand side is $\rho_{A'B'C'}^{1'}$, which has only three components,

$$\rho_{1'2'1'}^{1'} = 1, \quad \rho_{2'1'1'}^{1'} = j^2, \quad \rho_{1'1'2'}^{1'} = j,$$

The invariance group of cubic matrices

- Now, $\rho_{121}^1 = 1$, and we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ -matrix on the right-hand side is $\rho_{A'B'C'}^{1'}$, which has only three components,

$$\rho_{1'2'1'}^{1'} = 1, \quad \rho_{2'1'1'}^{1'} = j^2, \quad \rho_{1'1'2'}^{1'} = j,$$

- which leads to the following equation:

$$\Lambda_1^{1'} = U_1^{1'} U_2^{2'} U_1^{1'} + j^2 U_1^{2'} U_2^{1'} U_1^{1'} + j U_1^{1'} U_2^{1'} U_2^{2'} = U_1^{1'} (U_2^{2'} U_1^{1'} - U_2^{1'} U_2^{2'}),$$

because $j^2 + j = -1$.

The invariance group of cubic matrices

For the alternative choice $\alpha' = 2'$ the ρ -matrix on the right-hand side is $\rho_{A'B'C'}^{2'}$, whose three non-vanishing components are

$$\rho_{2'1'2'}^{2'} = 1, \quad \rho_{1'2'2'}^{2'} = j^2, \quad \rho_{2'2'1'}^{2'} = j.$$

The corresponding equation becomes now:

$$\Lambda_1^{2'} = U_1^{2'} U_2^{1'} U_1^{2'} + j^2 U_1^{1'} U_2^{2'} U_1^{2'} + j U_1^{2'} U_2^{2'} U_1^{1'} = U_1^{2'} (U_2^{1'} U_1^{2'} - U_1^{1'} U_2^{2'}),$$

The invariance group of cubic matrices

The remaining two equations are obtained in a similar manner. We choose now the three lower indices on the left-hand side equal to another independent combination, (212). Then the ρ -matrix on the left hand side must be ρ^2 whose component ρ_{212}^2 is equal to 1. This leads to the following equation when $\alpha' = 1'$:

$$\Lambda_2^{1'} = U_2^{1'} U_1^{2'} U_2^{1'} + j^2 U_2^{2'} U_1^{1'} U_2^{1'} + j U_2^{1'} U_1^{1'} U_2^{2'} = U_2^{1'} (U_2^{1'} U_1^{2'} - U_1^{1'} U_2^{2'}),$$

and the fourth equation corresponding to $\alpha' = 2'$ is:

$$\Lambda_2^{2'} = U_2^{2'} U_1^{1'} U_2^{2'} + j^2 U_2^{1'} U_1^{2'} U_2^{2'} + j U_2^{2'} U_1^{2'} U_2^{1'} = U_2^{2'} (U_1^{1'} U_2^{2'} - U_1^{2'} U_2^{1'}).$$

The invariance group of cubic matrices

The determinant of the 2×2 complex matrix $U_B^{A'}$ appears everywhere on the right-hand side.

$$\Lambda_1^{2'} = -U_1^{2'} [\det(U)], \quad (35)$$

The remaining two equations are obtained in a similar manner, resulting in the following:

$$\Lambda_2^{1'} = -U_2^{1'} [\det(U)], \quad \Lambda_2^{2'} = U_2^{2'} [\det(U)]. \quad (36)$$

The determinant of the 2×2 complex matrix $U_B^{A'}$ appears everywhere on the right-hand side. Taking the determinant of the matrix $\Lambda_\beta^{\alpha'}$ one gets immediately

$$\det(\Lambda) = [\det(U)]^3. \quad (37)$$

The invariance group of cubic matrices

However, the U -matrices on the right-hand side are defined only up to the phase, which due to the cubic character of the covariance relations and they can take on three different values: 1, j or j^2 , i.e. the matrices $j U_B^{A'}$ or $j^2 U_B^{A'}$ satisfy the same relations as the matrices $U_B^{A'}$ defined above. The determinant of U can take on the values 1, j or j^2 if $\det(\Lambda) = 1$

But for the time being, there we have no reason yet to impose the unitarity condition. It can be derived from the conditions imposed on the invariance of binary relations between θ^A and their conjugates $\bar{\theta}^{\dot{B}}$.

The vector representation

A similar covariance requirement can be formulated with respect to the set of 2-forms mapping the quadratic quark-anti-quark combinations into a four-dimensional linear real space. As we already saw, the symmetry (16) imposed on these expressions reduces their number to four. Let us define two quadratic forms, $\pi_{A\dot{B}}^\mu$ and its conjugate $\bar{\pi}_{\dot{B}A}^\mu$

$$\pi_{A\dot{B}}^\mu \theta^A \bar{\theta}^{\dot{B}} \quad \text{and} \quad \bar{\pi}_{\dot{B}A}^\mu \bar{\theta}^{\dot{B}} \theta^A. \quad (38)$$

The Greek indices μ, ν, \dots take on four values, and we shall label them 0, 1, 2, 3.

The vector representation

The four tensors $\pi_{A\dot{B}}^\mu$ and their hermitina conjugates $\bar{\pi}_{\dot{B}A}^\mu$ define a bi-linear mapping from the product of quark and anti-quark cubic algebras into a linear four-dimensional vector space, whose structure is not yet defined.

Let us impose the following invariance condition:

$$\pi_{A\dot{B}}^\mu \theta^A \bar{\theta}^{\dot{B}} = \bar{\pi}_{\dot{B}A}^\mu \bar{\theta}^{\dot{B}} \theta^A. \quad (39)$$

The vector representation

It follows immediately from (16) that

$$\pi_{A\dot{B}}^{\mu} = -j^2 \bar{\pi}_{\dot{B}A}^{\mu}. \quad (40)$$

Such matrices are non-hermitian, and they can be realized by the following substitution:

$$\pi_{A\dot{B}}^{\mu} = j^2 i \sigma_{A\dot{B}}^{\mu}, \quad \bar{\pi}_{\dot{B}A}^{\mu} = -j i \sigma_{\dot{B}A}^{\mu} \quad (41)$$

where $\sigma_{A\dot{B}}^{\mu}$ are the unit 2 matrix for $\mu = 0$, and the three hermitian Pauli matrices for $\mu = 1, 2, 3$.

The vector representation

Again, we want to get the same form of these four matrices in another basis. Knowing that the lower indices A and \dot{B} undergo the transformation with matrices $U_B^{A'}$ and $\bar{U}_{\dot{B}}^{\dot{A}'}$, we demand that there exist some 4×4 matrices $\Lambda_{\nu}^{\mu'}$ representing the transformation of lower indices by the matrices U and \bar{U} :

$$\Lambda_{\nu}^{\mu'} \pi_{A\dot{B}}^{\nu} = U_A^{A'} \bar{U}_{\dot{B}}^{\dot{B}'} \pi_{A'\dot{B}'}^{\mu'}, \quad (42)$$

The vector representation

The first four equations relating the 4×4 real matrices $\Lambda_{\nu}^{\mu'}$ with the 2×2 complex matrices $U_B^{A'}$ and $\bar{U}_{\dot{B}}^{\dot{A}'}$ are as follows:

$$\Lambda_0^{0'} + \Lambda_3^{0'} = U_1^{1'} \bar{U}_{\dot{1}}^{\dot{1}'} + U_1^{2'} \bar{U}_{\dot{1}}^{\dot{2}'}$$

$$\Lambda_0^{0'} - \Lambda_3^{0'} = U_2^{1'} \bar{U}_{\dot{2}}^{\dot{1}'} + U_2^{2'} \bar{U}_{\dot{2}}^{\dot{2}'}$$

$$\Lambda_0^{0'} - i\Lambda_2^{0'} = U_1^{1'} \bar{U}_{\dot{2}}^{\dot{1}'} + U_1^{2'} \bar{U}_{\dot{2}}^{\dot{2}'}$$

$$\Lambda_0^{0'} + i\Lambda_2^{0'} = U_2^{1'} \bar{U}_{\dot{1}}^{\dot{1}'} + U_2^{2'} \bar{U}_{\dot{1}}^{\dot{2}'}$$

The vector representation

The next four equations relating the 4×4 real matrices $\Lambda_\nu^{\mu'}$ with the 2×2 complex matrices $U_B^{A'}$ and $\bar{U}_{\dot{B}}^{\dot{A}'}$ are as follows:

$$\Lambda_0^{1'} + \Lambda_3^{1'} = U_1^{1'} \bar{U}_{\dot{1}}^{\dot{2}'} + U_1^{2'} \bar{U}_{\dot{1}}^{\dot{1}'}$$

$$\Lambda_0^{1'} - \Lambda_3^{1'} = U_2^{1'} \bar{U}_{\dot{2}}^{\dot{2}'} + U_2^{2'} \bar{U}_{\dot{2}}^{\dot{1}'}$$

$$\Lambda_0^{1'} - i\Lambda_2^{1'} = U_1^{1'} \bar{U}_{\dot{2}}^{\dot{2}'} + U_1^{2'} \bar{U}_{\dot{2}}^{\dot{1}'}$$

$$\Lambda_0^{1'} + i\Lambda_2^{1'} = U_2^{1'} \bar{U}_{\dot{1}}^{\dot{2}'} + U_2^{2'} \bar{U}_{\dot{1}}^{\dot{1}'}$$

The vector representation

The next four equations relating the 4×4 real matrices $\Lambda_{\nu}^{\mu'}$ with the 2×2 complex matrices $U_B^{A'}$ and $\bar{U}_{\dot{B}}^{\dot{A}'}$ are as follows:

$$\Lambda_0^{2'} + \Lambda_3^{2'} = -i U_1^{1'} \bar{U}_1^{\dot{2}'} + i U_1^{2'} \bar{U}_1^{\dot{1}'}$$

$$\Lambda_0^{2'} - \Lambda_3^{2'} = -i U_2^{1'} \bar{U}_2^{\dot{2}'} + i U_2^{2'} \bar{U}_2^{\dot{1}'}$$

$$\Lambda_0^{2'} - i\Lambda_2^{2'} = -i U_1^{1'} \bar{U}_2^{\dot{2}'} + i U_1^{2'} \bar{U}_2^{\dot{1}'}$$

$$\Lambda_0^{2'} + i\Lambda_2^{2'} = -i U_2^{1'} \bar{U}_1^{\dot{2}'} + i U_2^{2'} \bar{U}_1^{\dot{1}'}$$

The vector representation

The last four equations relating the 4×4 real matrices $\Lambda_{\nu}^{\mu'}$ with the 2×2 complex matrices $U_B^{A'}$ and $\bar{U}_{\dot{B}}^{\dot{A}'}$ are as follows:

$$\Lambda_0^{3'} + \Lambda_3^{3'} = U_1^{1'} \bar{U}_{\dot{1}}^{\dot{1}'} - U_1^{2'} \bar{U}_{\dot{1}}^{\dot{2}'}$$

$$\Lambda_0^{3'} - \Lambda_3^{3'} = U_2^{1'} \bar{U}_{\dot{2}}^{\dot{1}'} - U_2^{2'} \bar{U}_{\dot{2}}^{\dot{2}'}$$

$$\Lambda_0^{3'} - i\Lambda_2^{3'} = U_1^{1'} \bar{U}_{\dot{2}}^{\dot{1}'} - U_1^{2'} \bar{U}_{\dot{2}}^{\dot{2}'}$$

$$\Lambda_0^{3'} + i\Lambda_2^{3'} = U_2^{1'} \bar{U}_{\dot{1}}^{\dot{1}'} - U_2^{2'} \bar{U}_{\dot{1}}^{\dot{2}'}$$

The metric tensor $g_{\mu\nu}$

With the invariant “spinorial metric” in two complex dimensions, ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$ such that $\varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}}$, we can define the contravariant components $\pi^\nu{}^{A\dot{B}}$. It is easy to show that the Minkowskian space-time metric, invariant under the Lorentz transformations, can be defined as

$$g^{\mu\nu} = \frac{1}{2} \left[\pi_{A\dot{B}}^\mu \pi^{\nu A\dot{B}} \right] = \text{diag}(+, -, -, -) \quad (43)$$

Together with the anti-commuting spinors ψ^α the four real coefficients defining a Lorentz vector, $x_\mu \pi_{A\dot{B}}^\mu$, can generate now the supersymmetry via standard definitions of super-derivations.

The invariance group of cubic matrices

Let us then choose the matrices $\Lambda_{\beta}^{\alpha'}$ to be the usual spinor representation of the $SL(2, \mathbf{C})$ group, while the matrices $U_B^{A'}$ will be defined as follows:

$$U_1^{1'} = j\Lambda_1^{1'}, U_2^{1'} = -j\Lambda_2^{1'}, U_1^{2'} = -j\Lambda_1^{2'}, U_2^{2'} = j\Lambda_2^{2'}, \quad (44)$$

the determinant of U being equal to j^2 .

The invariance group of cubic matrices

Obviously, the same reasoning leads to the conjugate cubic representation of the same symmetry group $SL(2, \mathbf{C})$ if we require the covariance of the conjugate tensor

$$\bar{\rho}_{\dot{D}\dot{E}\dot{F}}^{\dot{\beta}} = j \bar{\rho}_{\dot{E}\dot{F}\dot{D}}^{\dot{\beta}} = j^2 \bar{\rho}_{\dot{F}\dot{D}\dot{E}}^{\dot{\beta}},$$

by imposing the equation similar to (34)

$$\Lambda_{\dot{\beta}}^{\dot{\alpha}'} \bar{\rho}_{\dot{A}\dot{B}\dot{C}}^{\dot{\beta}} = \bar{\rho}_{\dot{A}'\dot{B}'\dot{C}'}^{\dot{\alpha}'} \bar{U}_{\dot{A}}^{\dot{A}'} \bar{U}_{\dot{B}}^{\dot{B}'} \bar{U}_{\dot{C}}^{\dot{C}'}. \quad (45)$$

The matrix \bar{U} is the complex conjugate of the matrix U , and its determinant is equal to j .

The vector representation

Moreover, the two-component entities obtained as images of cubic combinations of quarks, $\psi^\alpha = \rho_{ABC}^\alpha \theta^A \theta^B \theta^C$ and $\bar{\psi}^{\dot{\beta}} = \bar{\rho}_{\dot{D}\dot{E}\dot{F}}^{\dot{\beta}} \bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}}$ should anti-commute, because their arguments do so, by virtue of (16):

$$(\theta^A \theta^B \theta^C)(\bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}}) = -(\bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}})(\theta^A \theta^B \theta^C)$$

$SL(2, \mathbb{C})$ group conserves the ternary algebra

We have found the way to derive the covering group of the Lorentz group acting on spinors via the usual spinorial representation. The spinors are obtained as the homomorphic image of tri-linear combination of three quarks (or anti-quarks). The quarks transform with matrices U (or \bar{U} for the anti-quarks), but these matrices are not unitary: their determinants are equal to j^2 or j , respectively. So, quarks cannot be put on the same footing as classical spinors; they transform under a Z_3 -covering of the $SL(2, \mathbb{C})$ group.

Fractional electric charge

- In the spirit of the Kaluza-Klein theory, the electric charge of a particle is the eigenvalue of the fifth component of the generalized momentum operator:

$$\hat{p}_5 = -i\hbar \frac{\partial}{\partial x^5},$$

where x^5 stays for the fifth coordinate.

Fractional electric charge

- ▶ In the spirit of the Kaluza-Klein theory, the electric charge of a particle is the eigenvalue of the fifth component of the generalized momentum operator:

$$\hat{p}_5 = -i\hbar \frac{\partial}{\partial x^5},$$

where x^5 stays for the fifth coordinate.

- ▶ Let the observed electric charge of the proton be e and that of the electron $-e$. If we put now the following factors multiplying the generators θ^1 and θ^2 :

$$\Theta^1 = \theta^1 e^{-\frac{iqx^5}{3\hbar}}, \quad \Theta^2 = \theta^2 e^{\frac{2iqx^5}{3\hbar}},$$

Fractional electric charge

- ▶ The eigenvalues of the fifth component of the momentum operator are, respectively:

$$\hat{p}\Theta^1 = -i\hbar\partial_5(\theta^1 e^{\frac{2iqx^5}{3\hbar}}) = -\frac{q}{3}\Theta^1,$$

$$\hat{p}\Theta^2 = -i\hbar\partial_5(\theta^2 e^{-\frac{iqx^5}{3\hbar}}) = \frac{2q}{3}\Theta^2$$

Fractional electric charge

- ▶ The eigenvalues of the fifth component of the momentum operator are, respectively:

$$\hat{p}\Theta^1 = -i\hbar\partial_5(\theta^1 e^{\frac{2iqx^5}{3\hbar}}) = -\frac{q}{3}\Theta^1,$$

$$\hat{p}\Theta^2 = -i\hbar\partial_5(\theta^2 e^{-\frac{iqx^5}{3\hbar}}) = \frac{2q}{3}\Theta^2$$

- ▶ The only non-vanishing products of our generators being $\theta^1\theta^1\theta^2$ and $\theta^1\theta^2\theta^2$, for the admissible products of functions representing the ternary combinations we readily get:

$$\hat{p}\theta^1\theta^1\theta^2 = q\theta^1\theta^1\theta^2, \quad \hat{p}\theta^1\theta^2\theta^2 = 0,$$

which correspond to the usual combinations of **(udd)** and **(udd)** quarks, representing two baryons: the proton and the

Three generators of the Z_3 -graded algebra

Consider now three generators, Q^a , $a = 1, 2, 3$, and their conjugates $\bar{Q}^{\dot{b}}$ satisfying similar cubic commutation relations as in the two-dimensional case:

$$Q^a Q^b Q^c = j Q^b Q^c Q^a = j^2 Q^c Q^a Q^b,$$

$$\bar{Q}^{\dot{a}} \bar{Q}^{\dot{b}} \bar{Q}^{\dot{c}} = j^2 \bar{Q}^{\dot{b}} \bar{Q}^{\dot{c}} \bar{Q}^{\dot{a}} = j \bar{Q}^{\dot{c}} \bar{Q}^{\dot{a}} \bar{Q}^{\dot{b}},$$

$$Q^a \bar{Q}^{\dot{b}} = -j \bar{Q}^{\dot{b}} Q^a.$$

With the indices a, b, c, \dots ranging from 1 to 3 we get *eight* linearly independent combinations of three undotted indices, and the same number of combinations of dotted ones.

Three generators

They can be arranged as follows:

$$\begin{aligned} &Q^3 Q^2 Q^3, \quad Q^2 Q^3 Q^2, \quad Q^1 Q^2 Q^1, \\ &Q^3 Q^1 Q^3, \quad Q^1 Q^2 Q^1, \quad Q^2 Q^1 Q^2, \\ &Q^1 Q^2 Q^3, \quad Q^3 Q^2 Q^1, \end{aligned}$$

while the quadratic expressions of grade 0, $Q^a \bar{Q}^b$ span a 9-dimensional subspace in the finite algebra generated by Q^a 's.

Cubic matrices in three dimensions

The invariant 3-form mapping these combinations onto some eight-dimensional space must have also eight independent components (over real numbers). One can easily define these three-dimensional “cubic matrices” as follows:

$$K_{121}^{3+} = j \quad K_{112}^{3+} = j^2 \quad K_{211}^{3+} = 1; \quad K_{212}^{3-} = j \quad K_{221}^{3-} = j^2 \quad K_{122}^{3-} = 1;$$

$$K_{313}^{2+} = j \quad K_{331}^{2+} = j^2 \quad K_{133}^{2+} = 1; \quad K_{131}^{2-} = j \quad K_{113}^{2-} = j^2 \quad K_{311}^{2-} = 1;$$

$$K_{232}^{1+} = j \quad K_{223}^{1+} = j^2 \quad K_{322}^{1+} = 1; \quad K_{323}^{1-} = j \quad K_{332}^{1-} = j^2 \quad K_{233}^{1-} = 1;$$

$$K_{123}^7 = j \quad K_{231}^7 = j^2 \quad K_{312}^7 = 1, \quad K_{132}^8 = j \quad K_{321}^8 = j^2 \quad K_{213}^8 = 1,$$

all other components being identically zero.

Cubic matrices in three dimensions

The structure of the set of cubic K -matrices is similar to the structure of the root diagram of the Lie algebra of the $SU(3)$ group.

We have three pairs of generators behaving like the three $SU(2)$ subgroups, K^{3+} , K^{3-} , K^{1+} , K^{1-} , K^{2+} , K^{2-} and two extra generators behaving like the Cartan subalgebra of the $SU(3)$ Lie algebra, K^7 and K^8 .

Cubic matrices in three dimensions

A similar covariance requirement can be applied to the K cubic matrices; We may ask for the following relation to be held:

$$\Lambda_{\beta}^{\alpha'} K_{abc}^{\beta} = U_a^{a'} U_b^{b'} U_c^{c'} K_{a'b'c'}^{\alpha'}$$

where the indices $\alpha, \beta \dots$ run from 1 to 8, and the indices a, b, c run from 1 to 3.

We have checked (with S. Shitov) that with these relations,

$$\det(\Lambda) = [\det(U)]^8$$

The matrices U are the fundamental representation of $SU(3)$, while the matrices Λ are the adjoint representation of $SU(3)$.

Cubic generalization of Dirac's equation

A ternary generalization of Clifford algebras. Instead of the usual binary relation defining the usual Clifford algebra,

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \mathbf{1}, \quad \text{with } g^{\mu\nu} - g^{\nu\mu} = 0$$

we should introduce its ternary generalization, which is quite obvious

$$Q^a Q^b Q^c + Q^b Q^c Q^a + Q^c Q^b Q^a = 3 \eta^{abc} \mathbf{1}, \quad (46)$$

where the tensor η^{abc} must satisfy

$$\eta^{abc} = \eta^{bca} = \eta^{cab}$$

Matrix representation of cubic Clifford algebra

The lowest-dimensional representation of such an algebra is given by complex 3×3 matrices:

$$Q^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \quad Q^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

These matrices are given the Z_3 -grade 1; their hermitian conjugates $Q^{*a} = (Q^a)^\dagger$ are of Z_3 -grade 2, whereas the diagonal matrices are of Z_3 -grade 0; it is easy to verify that so defined grades add up modulo 3.

The matrices Q^a ($a = 1, 2, 3$) satisfy the ternary relations

$$Q^a Q^b Q^c + Q^b Q^c Q^a + Q^c Q^a Q^b = 3 \eta^{abc} \mathbf{1}, \quad (47)$$

with η^{abc} a totally symmetric tensor, whose only non-vanishing components are

$$\eta^{111} = \eta^{222} = \eta^{333} = 1, \quad \eta^{123} = \eta^{231} \eta^{321} = j^2,$$

$$\text{and } \eta^{321} = \eta^{213} = \eta^{132} = j.$$

The cubic generalization of Dirac's equation

Therefore, the Z_3 -graded generalization of Dirac's equation should read:

$$\frac{\partial \psi}{\partial t} = Q^1 \frac{\partial \psi}{\partial x} + Q^2 \frac{\partial \psi}{\partial y} + Q^3 \frac{\partial \psi}{\partial z} + B m \psi \quad (48)$$

where ψ stands for a triplet of wave functions, which can be considered either as a column, or as a grade 1 matrix with three non-vanishing entries $u \ v \ w$, and B is the diagonal 3×3 matrix with the eigenvalues $1 \ j$ and j^2 . It is interesting to note that this is possible only with *three* spatial coordinates.

The independent solutions of cubic equation

In order to diagonalize this equation, we must act three times with the same operator, which will lead to the same equation of *third order*, satisfied by each of the three components u , v , w , e.g.:

$$\frac{\partial^3 u}{\partial t^3} = \left[\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3} - 3 \frac{\partial^3}{\partial x \partial y \partial z} \right] u + m^3 u \quad (49)$$

This equation can be solved by separation of variables; the time-dependent and the space-dependent factors have the same structure:

$$A_1 e^{\omega t} + A_2 e^{j\omega t} + A_3 e^{j^2\omega t}, \quad B_1 e^{\mathbf{k} \cdot \mathbf{r}} + B_2 e^{j\mathbf{k} \cdot \mathbf{r}} + B_3 e^{j^2\mathbf{k} \cdot \mathbf{r}}$$

Cubic generalization of Dirac's equation

The independent solutions of the third-order equation can be arranged in a 3×3 matrix as follows:

$$\begin{pmatrix} A_{11} e^{\omega t + \mathbf{k} \cdot \mathbf{r}} & A_{12} e^{\omega t - \frac{\mathbf{k} \cdot \mathbf{r}}{2}} \cos \xi & A_{13} e^{\omega t - \frac{\mathbf{k} \cdot \mathbf{r}}{2}} \sin \xi \\ A_{21} e^{-\frac{\omega t}{2} + \mathbf{k} \cdot \mathbf{r}} \cos \tau & A_{22} e^{-\frac{\omega t}{2} - \frac{\mathbf{k} \cdot \mathbf{r}}{2}} \cos \tau \cos \xi & A_{23} e^{-\frac{\omega t}{2} - \frac{\mathbf{k} \cdot \mathbf{r}}{2}} \cos \tau \sin \xi \\ A_{31} e^{-\frac{\omega t}{2} + \mathbf{k} \cdot \mathbf{r}} \sin \tau & A_{32} e^{-\frac{\omega t}{2} - \frac{\mathbf{k} \cdot \mathbf{r}}{2}} \sin \tau \cos \xi & A_{33} e^{-\frac{\omega t}{2} - \frac{\mathbf{k} \cdot \mathbf{r}}{2}} \sin \tau \sin \xi \end{pmatrix}$$

where $\tau = \frac{\sqrt{3}}{2} \omega t$ and $\xi = \frac{\sqrt{3}}{2} \mathbf{k} \cdot \mathbf{r}$. The parameters ω , \mathbf{k} and m must satisfy the cubic dispersion relation:

$$\omega^3 = k_x^3 + k_y^3 + k_z^3 - 3 k_x k_y k_z + m^3$$

Cubic generalization of Dirac's equation

This relation is invariant under the simultaneous change of sign of ω , \mathbf{k} and m , which suggests the introduction of another set of solutions constructed in the same manner, but with minus sign in front of ω (or \mathbf{k}), which we shall call *conjugate* solutions.

Although neither of these functions belongs to the space of tempered distributions, on which a Fourier transform can be performed, their ternary skew-symmetric products contain only trigonometric functions, depending on the combinations $2(\tau - \xi)$ and $2(\tau + \xi)$.

The same is true for the binary products of “conjugate” solutions, with the opposite signs of ωt and $\mathbf{k} \cdot \mathbf{r}$ in the exponentials.

Cubic generalization of Dirac's equation

However, taking at random three independent solutions of or cubic equation, with three different sets of frequencies and wave vectors, say

$$(\omega_1, \mathbf{k}_1), \quad (\omega_2, \mathbf{k}_2), \quad (\omega_3, \mathbf{k}_3),$$

and taking their ternary products will not lead to the cancellation of non-propagating, real exponential terms, even if all three sets satisfy the required cubic dispersion relation

$$\omega^2 = k_x^3 + k_y^3 + k_z^3 - 3 k_x k_y k_z + m^3$$

Cubic generalization of Dirac's equation

- This fact suggests that it is possible to obtain via linear combinations of these products the solutions of *second* or *first order* differential equations, like Klein-Gordon or Dirac equation.

Cubic generalization of Dirac's equation

- ▶ This fact suggests that it is possible to obtain via linear combinations of these products the solutions of *second* or *first order* differential equations, like Klein-Gordon or Dirac equation.
- ▶ Still, the parameters ω and k do not satisfy the proper mass shell relations; however, it is possible to find new parameters, which are linear combinations of these, that will satisfy quadratic relations that may be interpreted as a mass shell equation.

Cubic generalization of Dirac's equation

We can more readily see this if we use the following parametrisation: let us put

$$\zeta = (k_x + k_y + k_z), \quad \chi = \operatorname{Re}(jk_x + j^2 k_y + k_z), \quad \eta = \operatorname{Im}(jk_x + j^2 k_y + k_z),$$

$$\text{and} \quad r^2 = \chi^2 + \eta^2 \quad \phi = \operatorname{Arctg}(\eta/\chi).$$

In these coordinates the cubic mass hyperboloid equation becomes

$$\omega^3 - \zeta r^2 = m^3 \tag{50}$$

Cubic generalization of Dirac's equation

Two obvious symmetries can be immediately seen here, the rotation around the axis $[1, 1, 1]$ ($\phi \rightarrow \phi + \delta\phi$), and simultaneous dilatation of ζ and r :

$$r \rightarrow \lambda r, \quad \zeta \rightarrow \lambda^{-2} \zeta$$

The same relation can be factorized as

$$(\omega + \zeta)(\omega^2 - r^2) + (\omega - \zeta)(\omega^2 + r^2) = 2m^3$$

We can define a one-dimensional subset of the above 3-dimensional hypersurface by requiring

$$\omega^2 - r^2 = [2m^3 - (\omega - \zeta)(\omega^2 + r^2)]/(\omega + \zeta) = M^2 = \text{Const.}$$

Cubic generalization of Dirac's equation

If we have *three* hypersurfaces (corresponding to the dispersion relations of three quarks satisfying the 3-rd order differential equation), which are embedded in the 12-dimensional space $M_4 \times M_4 \times M_4$, then the resulting 3-dimensional hypersurface defined by the above constrained applied to each of the three dispersion relations independently will produce the ordinary mass hyperboloid

$$\omega_1^2 + \omega_2^2 + \omega_3^2 - r_1^2 - r_2^2 - r_3^2 = \Omega^2 - r_1^2 - r_2^2 - r_3^2 = 3 M^2$$

Cubic generalization of Dirac's equation

- ▶ Therefore, extra conditions should exist that would impose the same values of ω and \mathbf{k} for the three solutions whose product should represent a propagating function.

Cubic generalization of Dirac's equation

- ▶ Therefore, extra conditions should exist that would impose the same values of ω and \mathbf{k} for the three solutions whose product should represent a propagating function.
- ▶ This reminds the situation with two interdependent fields, the electric field \mathbf{E} and the magnetic field \mathbf{B} . They both satisfy the d'Alembert equation *in vacuo*, supposed they satisfy also $\text{div}\mathbf{E} = 0$, $\text{div}\mathbf{B} = 0$:

$$\square \mathbf{E} = 0, \quad \square \mathbf{B} = 0.$$

and one could in principle choose two solutions with two different wave vectors and frequencies:

$$\mathbf{E} = \mathbf{E}_0 e^{i(\omega_1 t - \mathbf{k}_1 \cdot \mathbf{r})}, \quad \mathbf{B} = \mathbf{B}_0 e^{i(\omega_2 t - \mathbf{k}_2 \cdot \mathbf{r})}.$$

Cubic generalization of Dirac's equation

In fact, both fields must satisfy not only the d'Alembert equation, but also the original Maxwell equations of the *first order* :

$$\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \text{rot} \mathbf{B},$$

$$-\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \text{rot} \mathbf{E},$$

which shows immediately that the solution in form of a wave must display the same exponential factor for both fields, with common frequency ω and the same wave vector \mathbf{k} .

Cubic generalization of Dirac's equation

The two first order equations for \mathbf{E} and \mathbf{B} differ only by the sign of the time derivative, and can be written in a matrix form as

$$\frac{1}{c} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{rot} \\ -\mathbf{rot} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$$

the two matrices realizing two possible representations of the non-trivial element of the Z_2 group.

Cubic generalization of Dirac's equation

There exists a **quadratic** combination of fields **E** and **B** satisfying a first-order equation:

$$\frac{1}{c} \frac{\partial}{\partial t} \left[\frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) \right] + \text{div}(\mathbf{E} \wedge \mathbf{B}) = 0.$$

Cubic generalization of Dirac's equation

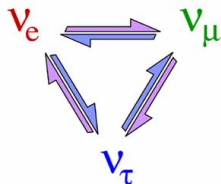
In the same spirit, one may imagine a three-component generalization involving the Z_3 group instead of Z_2 . With three components, a similar set of three equations would be:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \frac{1}{c} \frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \chi \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & \hat{H} & 0 \\ 0 & 0 & j\hat{H} \\ j^2\hat{H} & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \\ \xi \end{pmatrix}$$

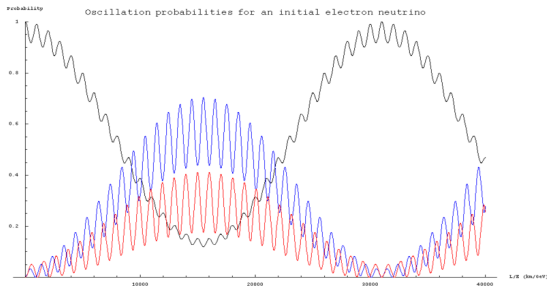
where the operators \hat{H}_{ab} , $a, b = 1, 2, 3$ must not be exactly the same (they can correspond to different masses, for example). In order to include somehow the Lorentz group, they should most probably contain the Pauli operators

$$\vec{\sigma} \cdot \vec{grad}$$

Example: the neutrino oscillations



The periodic change of neutrino flavor from one type into another is referred to as neutrino oscillations.



The neutrino oscillations mix up three types of particles, each of different lepton family.

They propagate simultaneously according to the cubic Dirac equation.