Curved Momentum Space and Relative Locality

XXIX Max Born Symposium, 2011
Dedicated to Jurek Lukierski on his 75th birthday
Based on the papers

- L. Freidel, L. Smolin „Gamma ray burst delay times probe the geometry of momentum space” arXiv:1103.5626 [hep-th]
- And more to come ...
Plan

• Why curved momentum space?
• Geometry of momentum space and particle action.
• Geometry of kappa-momentum space.
• Relative locality.
Why curved momentum space?

- The assumption that the angle sum is less than $180^\circ$ leads to a geometry quite different from Euclid’s. It depends on a constant, which is not given a priori. As a joke I even wished Euclidean geometry was not true, for then we would have an absolute measure of length a priori.

- If a fundamental scale is given, the space in question may have nontrivial geometry.
• Given a scale we can introduce nonlinear geometric structures.

• For example: given velocity scale \( c \) we can introduce a non-trivial geometry on the velocity space and that's exactly what we do in SR.

• We implement this by a non-trivial rule of velocity addition

\[
\vec{v} \oplus \vec{u} = \frac{1}{1 + \vec{v}\vec{u}/c^2} \left( \vec{v} + \frac{\vec{u}}{\gamma_v} + \frac{1}{c^2} \frac{\gamma_v}{1 + \gamma_v} (\vec{v}\vec{u})\vec{v} \right)
\]

• which is neither symmetric nor associative.
The momentum scale

• In $2+1$ dimensions $G_N$ has the dimension of inverse mass and serves as a momentum space scale.

• In this case the effective theory of particle’s kinematics has curved momentum space with the curvature scale $(G_N)^2$. 
**3+1 dimensions**

- In 3+1 dimension the scale might be provided by a particular semiclassical, flat spacetime limit of Quantum Gravity

\[ \hbar \to 0, G_N \to 0, \quad \text{but such that} \quad \sqrt{\frac{\hbar}{G_N}} \to \kappa \text{ is finite} \]

- but there might be other ways/.regimes making this scale appear.
And besides ...

- It was Max Born who (as it seems) was first to consider curved momentum space, in his paper of 1938.
The momentum space

• The properties of momentum space:
  1. It must be a manifold, so that the coordinates of points (four)-momenta are well defined;
  2. There is a special point – zero momentum – that is both diffeo and Lorentz invariant;
  3. It must possess metric and connection.
Particle kinematics

• To define particle’s kinematics one has to determine
  a. How free particles move: dispersion relation in vacuum;
Dispersion relation

• The dispersion relation is defined as a square of the distance from zero to the point $P$, with coordinates $p_{\mu}(P)$:

$$C(p) = D^2(p) - m^2$$

$$D^2(p) = \int ds \ g^{\mu\nu} \ p_\mu \ p_\nu \bigg|_{\text{geodesic}}$$

• We need metric to define dispersion relation!
**Free particle action**

- The action of a free relativistic particle with mass $m$ reads

\[ S^\text{free} = \int d\tau \, \dot{p}_\mu x^\mu + NC(p) \]

- with the standard equations of motion

\[ C(p) = 0, \quad \dot{p}_\mu = 0 \]

\[ \dot{x}^\mu = N \frac{\partial C(p)}{\partial p_\mu} \]
Interactions - qualitative picture

• Particles interact in vertices, where momentum conservation is imposed.

• Interacting particle’s worldlines are semi-infinite; for incoming particles worldlines $\tau \in (-\infty, 0)$, for outgoing $\tau \in (0, \infty)$. 
Vertices

Two-valent vertex

\[ K \equiv p \oplus q = 0 \]

Three-valent vertex

\[ K \equiv (p \oplus r) \ominus q = 0 \]

\[ \ominus : p \oplus (\ominus p) \equiv p \ominus p = 0 \]
Momentum addition

- In order to add two momenta the notion of connection is needed.

\[ p \oplus q \]
Connection

And vice versa, the composition law equips the momentum manifold with connection:

\[
(p \oplus q)_\mu = p_\mu + q_\mu - \Gamma^{\alpha\beta}_\mu p_\alpha q_\beta + \ldots
\]

\[
\Gamma^{\alpha\beta}_\mu = - \frac{\partial}{\partial p_\alpha} \frac{\partial}{\partial q_\beta} (p \oplus q)_\mu \bigg|_{p,q=0}
\]
Example: Kappa-Poincare

- In this case the momentum space is a group manifold of $AN(3)$.

$$AN(3) \in g(p) = \exp(iX^i p_i) \exp(iX^0 p_0)$$

$$[X^0, X^i] = -\frac{i}{\kappa} X^i , \quad [X^i, X^j] = 0$$
Example: Kappa-Poincare

• The group composition rule defines the composition rule of momenta.
  \[ g(p) g(q) = g(p \oplus q) \]

  \[
  (p \oplus q)_o = p_o + q_o \quad ; \quad (p \oplus q)_i = p_i + e^{-p_o/\kappa} q_i
  
  (\ominus p)_o = -p_o \quad ; \quad (\ominus p)_i = -e^{p_o/\kappa} p_i
  \]

• It is not symmetric, but associative.
Example: Kappa-Poincare

- There is a natural metric on AN(3) (the metric on de Sitter in flat coordinates), and one can compute the dispersion relation as a geodesic distance explicitly. As a result one gets a function of the standard Casimir of Kappa-Poincare.
Connection

• In general this connection is neither metric

\[ \nabla_{\mu} g^{\nu\rho} \neq 0 \]

• Nor torsion-free

\[ \Gamma_{\mu}^{\alpha\beta} \neq \Gamma_{\mu}^{\beta\alpha} \]
Example: Kappa-Poincare

- The connection on Kappa-Poincare momentum space is torsion-full but curvature-free.
Back to the action

\[ S_{\text{free}} = \sum_{i} \int_{\text{in/out}} d\tau \dot{p}^I_\mu x_i^\mu - N'C(p^I) \]

• At \( \tau=0 \) we impose the interaction vertex

\[ S_{\text{int}} = z^\mu K_{\mu}(p, q, r) \]

\[ K_{\mu} = [(p \oplus q) \ominus r]_{\mu} \]

• \( z^\mu \) is a Lagrange multiplier that imposes the momentum conservation.
Equations of motion

• For each semi-infinite wordline

\[ C(p') = 0, \quad \dot{p}'_\mu = 0 \]

\[ \dot{x'}_\mu = N' \frac{\partial C'(p')}{\partial p'_\mu} \]

• At the vertex

\[ K_\mu = 0, \quad x'_\mu(0) = z^v \frac{\partial K^v}{\partial p'_\mu} = U^{\mu}_{/v}(p) z^v \]
Relative locality

- The action is invariant under translations generated by the momentum composition law $K$:

\[ \delta x^\mu_i = \epsilon^\nu \frac{\partial K^\nu}{\partial p^i_\mu} = \epsilon^\nu \{ x^\mu_i, K^\nu \} \]

- $\delta x$ is momentum-independent if only if the momentum composition law is linear.
Relative locality

• If $K$ is a nonlinear function of the incoming/outgoing momenta the translation of the worldline depends on the momenta of all wordlines meeting in the event.

• Notice that the interaction coordinate $z^\mu$ is being translated by a constant vector $\varepsilon^\mu$ as it should be.
Relative locality

Local observer

Translated observer
Conclusion

• This is all very beautiful and exciting, but is it real?

• It is for the experiment to decide!