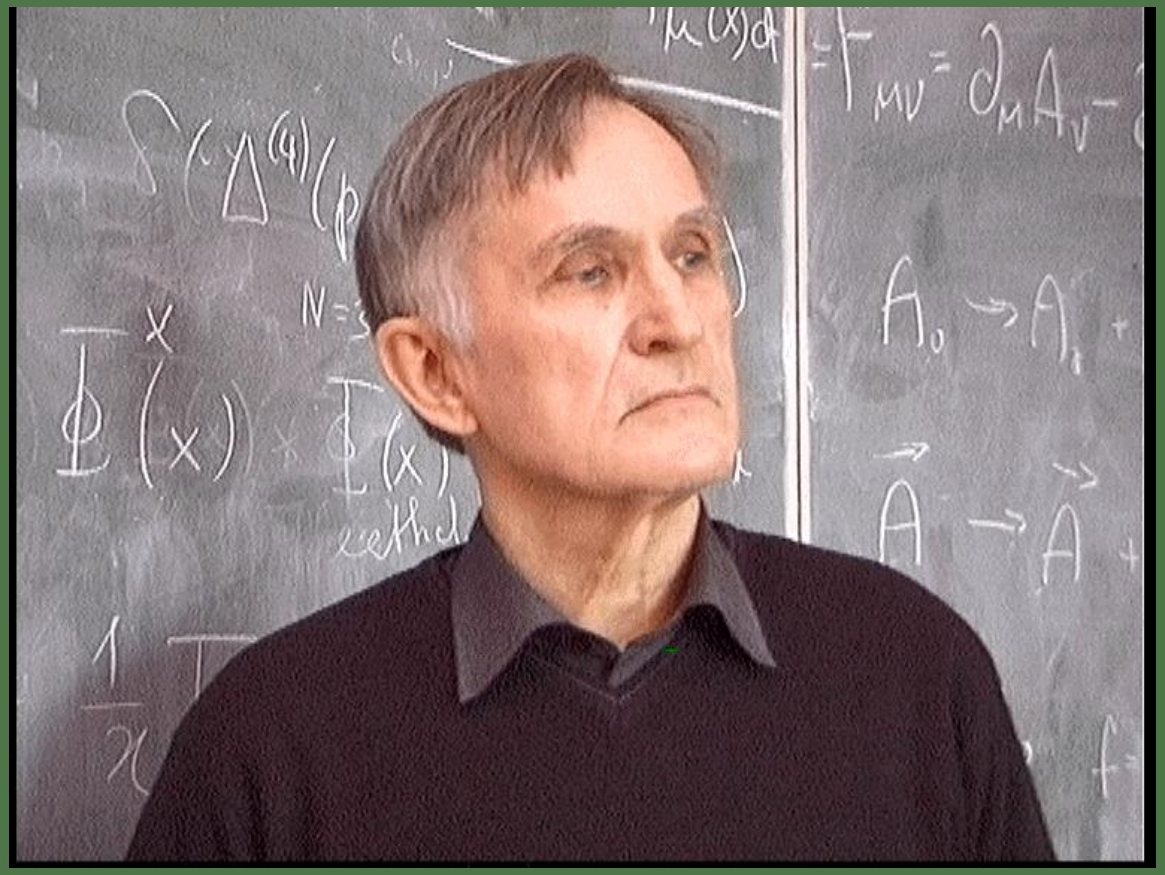
Curved Momentum Space and Relative Locality

XXIX Max Born Symposium, 2011

Dedicated to Jurek Lukierski on his 75th birthday



Based on the papers

- G. Amelino-Camelia, L. Freidel, JKG, L. Smolin ,,The principle of relative locality'' **arXiv:1101.0931** [hep-th]
- L. Freidel, L. Smolin ,,Gamma ray burst delay times probe the geometry of momentum space'' arXiv:1103.5626 [hep-th]
- G. Amelino-Camelia, L. Freidel, JKG, L. Smolin ,,Relative locality and the soccer ball problem'' arXiv:1104.2019 [hep-th]
- G. Amelino-Camelia, L. Freidel, JKG, L. Smolin ,,Relative locality: A deepening of the relativity principle'' **arXiv:1106.0313** [hep-th] (2nd prize in GRF 2011 essay competition)
- And more to come …

Plan

- Why curved momentum space?
- Geometry of momentum space and particle action.
- Geometry of kappa-momentum space.
- Relative locality.

Why curved momentum space?

- The assumption that the angle sum is less than 180° leads to a geometry quite different from Euclid's. It depends on a constant, which is not given a priori. As a joke I even wished Euclidean geometry was not true, for then we would have an absolute measure of length a priori.
- **If a fundamental scale is given, the space in question may have nontrivial geometry.**

- Given a scale we can introduce nonlinear geometric structures.
- For example: given velocity scale c we can introduce a non-trivial geometry on the velocity space and that's exactly what we do in SR.
- We implement this by a non-trivial rule of velocity addition

$$\vec{v} \oplus \vec{u} = \frac{1}{1 + \vec{v}\vec{u}/c^2}\left(\vec{v} + \frac{\vec{u}}{\gamma_v} + \frac{1}{c^2}\frac{\gamma_v}{1+\gamma_v}(\vec{v}\vec{u})\vec{v}\right)$$

- which is neither symmetric nor associative.

The momentum scale

- In 2+1 dimensions G_N has the dimension of inverse mass and serves as a momentum space scale.
- In this case the effective theory of particle's kinematics has curved momentum space with the curvature scale $(G_N)^2$.

3+1 dimensions

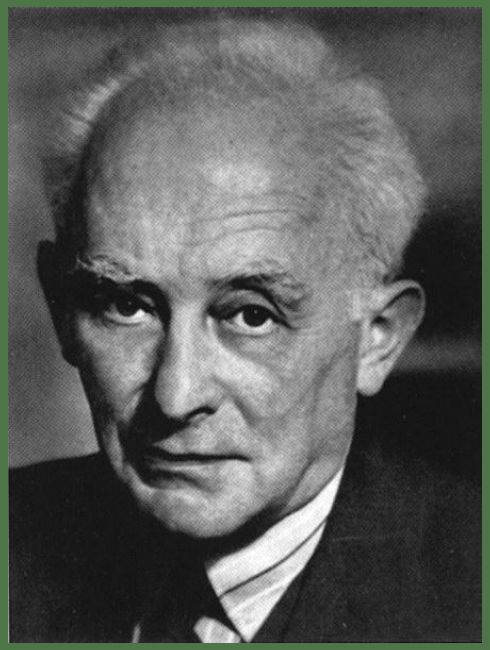
- In 3+1 dimension the scale might be provided by a particular semiclassical, flat spacetime limit of Quantum Gravity

$$\hbar \to 0,\ G_N \to 0,\ \text{but such that } \sqrt{\frac{\hbar}{G_N}} \to \kappa \text{ is finite}$$

- but there might be other ways/regimes making this scale appear.

And besides ...

- It was Max Born who (as it seems) was first to consider curved momentum space, in his paper of 1938.



The momentum space

- The properties of momentum space:
  1. It must be a manifold, so that the coordinates of points (four)-momenta are well defined;
  2. There is a special point – zero momentum – that is both diffeo and Lorentz invariant;
  3. It must possess metric and connection.

Particle kinematics

- To define particle's kinematics one has to determine
  a. How free particles move: dispersion relation in vacuum;
  b. How particles interact (locally): composition law for (four)-momenta.

Dispersion relation

- The dispersion relation is defined as a square of the distance from zero to the point P, with coordinates $p_\mu(P)$:

$$C(p) = D^2(p) - m^2$$

$$D^2(p) = \int_0^p ds\, g^{\mu\nu}\, \dot{p}_\mu \dot{p}_\nu \Big|_{geodesic}$$

- We need metric to define dispersion relation !

Free particle action

- The action of a free relativistic particle with mass m reads

$$\boxed{S^{free} = \int d\tau\, \dot{p}_\mu x^\mu + NC(p)}$$

- with the standard equations of motion

$$C(p) = 0, \qquad \dot{p}_\mu = 0$$

$$\dot{x}^\mu = N \frac{\partial C(p)}{\partial p_\mu}$$

Interactions – qualitative picture

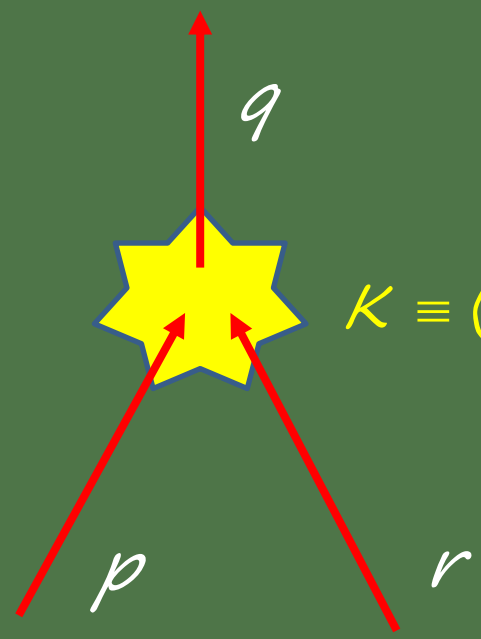
- Particles interact in vertices, where momentum conservation is imposed.
- Interacting particle's worldlines are semi-infinite; for incoming particles worldlines $\tau \in (-\infty, 0)$, for outgoing $\tau \in (0, \infty)$.

Vertices

Two-valent vertex

$$K \equiv p \ominus q = 0$$

Three-valent vertex

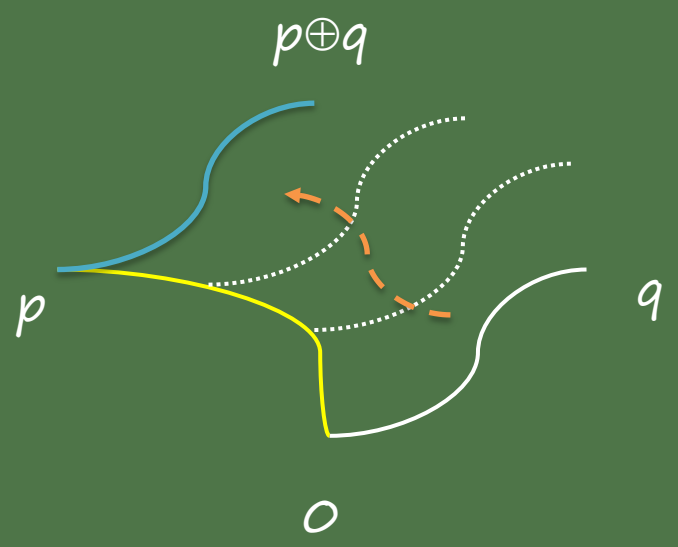


$$K \equiv (p \oplus r) \ominus q = 0$$

$$\ominus : p \oplus (\ominus p) \equiv p \ominus p = 0$$

Momentum addition

- In order to add two momenta the notion of connection is needed.



Connection

- And vice versa, the composition law equips the momentum manifold with connection:

$$(p \oplus q)_\mu = p_\mu + q_\mu - \Gamma_\mu{}^{\alpha\beta} p_\alpha q_\beta + \ldots$$

$$\Gamma_\mu{}^{\alpha\beta} = -\frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial q_\beta}(p \oplus q)_\mu \Big|_{p,q=0}$$

Example: Kappa-Poincare

- In this case the momentum space is a group manifold of AN(3).

$$AN(3) \in g(p) = \exp(iX^i p_i)\exp(iX^0 p_0)$$

$$[X^0, X^i] = -\frac{i}{\kappa} X^i, \quad [X^i, X^j] = 0$$

Example: Kappa-Poincare

- The group gomposition rule defines the composition rule of momenta.

$$g(p)\,g(q) = g(p \oplus q)$$

$$(p \oplus q)_0 = p_0 + q_0\,;(p \oplus q)_i = p_i + e^{-p_0/\kappa} q_i$$

$$(\ominus p)_0 = -p_0\,;(\ominus p)_i = -e^{p_0/\kappa} p_i$$

- It is not symmetric, but associative.

Example: Kappa-Poincare

- There is a natural metric on AN(3) (the metric on de Sitter in flat coordinates), and one can compute the dispersion relation as a geodesic distance explicitly. As a result one gets a function of the standard Casimir of Kappa-Poincare.

Connection

- In general this connection is neither metric

$$\nabla^{\mu}_{\Gamma} g^{\nu\rho} \neq 0$$

- Nor torsion-free

$$\Gamma_{\mu}{}^{\alpha\beta} \neq \Gamma_{\mu}{}^{\beta\alpha}$$

Example: Kappa-Poincare

- The connection on Kappa-Poincare momentum space is torsion-full but curvature-free.

Back to the action

$$S^{free} = \sum_{l} \int_{in/out} d\tau \, \dot{p}'_{\mu} x_{l}^{\mu} - N' C(p')$$

- At $\tau=0$ we impose the interaction vertex

$$S^{int} = z^{\mu} K_{\mu}(p,q,r)$$
$$K_{\mu} = [(p \oplus q) \ominus r]_{\mu}$$

- z^{μ} is a Lagrange multiplier that imposes the momentum conservation.

Equations of motion

- For each semi-infinite wordline

$$C(p^l) = 0, \qquad \dot{p}^l_\mu = 0$$

$$\dot{x}^\mu_l = N^l \frac{\partial C^l(p^l)}{\partial p^l_\mu}$$

- At the vertex

$$K_\mu = 0, \qquad x^\mu_l(0) = z^\nu \frac{\partial K_\nu}{\partial p^l_\mu} = U^\mu_{l\nu}(p)\, z^\nu$$

Relative locality

- The action is invariant under translations generated by the momentum composition law K:

$$\boxed{\delta x_I^\mu = \epsilon^\nu \frac{\partial K_\nu}{\partial p'_\mu} = \epsilon^\nu \left\{ x_I^\mu, K_\nu \right\}}$$

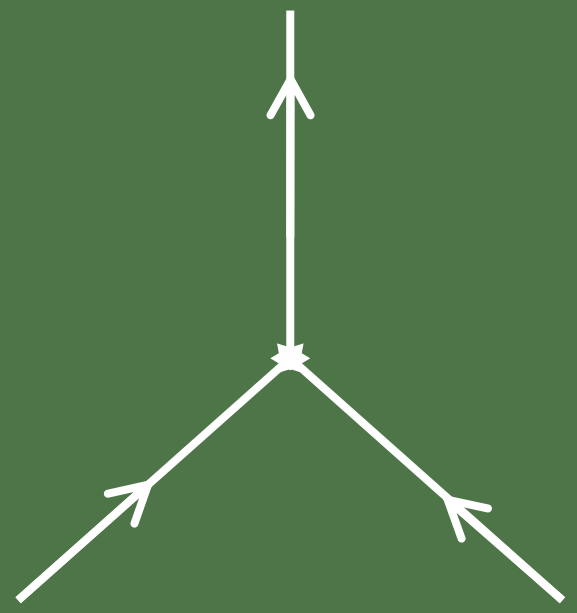
- δx is momentum-independent if only if the momentum composition law is linear.

Relative locality

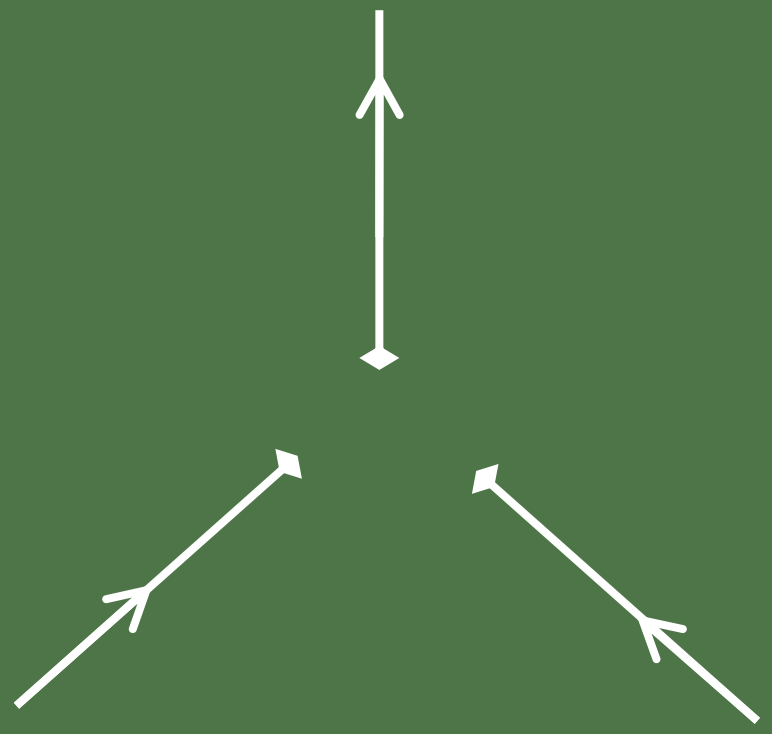
- If K is a nonlinear function of the incoming/outgoing momenta the translation of the worldline depends on the momenta of all wordlines meeting in the event.
- Notice that the interaction coordinate z^μ is being translated by a constant vector ε^μ as it should be.

Relative locality

Local observer



Translated observer



Conclusion

- This is all very beautiful and exciting, but is it real?
- It is for the experiment to decide!