Darkon fluid - a model for the dark sector of the Universe?

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Abstract
Introductory remarks
Galilean massless particles
Coupling to gravity
A self-gravitating fluid
- Lagrangian formulation
- Eulerian formulation
- Symmetries

Cosmological solutions
- Comparison with other models

Steady flow solutions
Final remarks
- Achievements
- Open problems

Abstract

We introduce darkons as fluid particles of a Galilean massless self-gravitating fluid. This fluid exhibits anisotropic scaling with $z = \frac{5}{3}$. The minimal gravitational coupling dynamically generates a gravitational mass density of either sign. Hence such fluid may serve as a model for the dark sector of the Universe. Its cosmological solutions give a deceleration phase for the early Universe and an acceleration phase for the late Universe. Will the steady flow solutions lead to a confining potential and so a possible model for halos?

Introductory remarks

1. Darkons: Fluid particles of a Galilean massless self-gravitating fluid with anisotropic scaling ($z = \frac{5}{3}$).
2. Some new nonrelativistic particles, responsible for the accelerated expansion of the Universe, must be massless.
(Massive particles always lead to attractive gravitation.)

Galilean massless particles

Task: Find the minimal dynamical realization of the unextended ($m = 0$) Galilean algebra in $d = 3$.

Solution: Consider the first order **Lagrangian** in 12-dim phase space

$$L_0 = p_i \dot{x}_i + q_i \dot{y}_i - p_i y_i$$

leading to the EOM

$$\dot{x}_i = y_i, \quad \dot{p}_i = 0, \quad \dot{q}_i = -p_i, \quad \text{and} \quad \dot{y}_i = 0.$$

Equivalent **Hamiltonian dynamics**

$$H = p_i y_i$$

with the nonvanishing Poisson-brackets (PBs)

$$[x_i, p_j] = \delta_{ij} \quad \text{and} \quad [y_i, q_j] = \delta_{ij}.$$

The boost generator K_i (by Noether thm.)

$$K_i = p_i t + q_i$$

leads to

$$[K_i, P_j] = 0$$

where $P_j = p_j$ is the generator of spatial translations.

L_0 is then the **minimal Lagrangian** giving this commutator.

Note: The unextended Galilean algebra may be enlarged by dilatations with arbitrary z. A further enlargement by expansions fixes $z = 2$.

Conclusion: Nonrelativistic massless particles possess a modified relation between energy and momentum (resp. velocity). They live in an enlarged phase space.

Coupling to gravity

Introduce minimal coupling to the gravitational force in accordance with **Einstein's equivalence principle** (free falling elevator)

As, at each fixed point $\vec{x}$ the gravitational force g_i is equivalent to acceleration $b_i(t)$, we note that the EOM

$$\ddot{x}_i = \dot{y}_i = 0$$

has to be modified to

$$\ddot{x}_i(t) = g_i(\vec{x}(t), t).$$

This equation can be realised if we add to L_0 an **interaction part** (minimal coupling)

$$L_{int} = -q_i g_i .$$

Then the EOM

$$\ddot{q}_i = -\dot{p}_i = 0$$

gets replaced by

$$\ddot{q}_i = q_k \, \partial_i g_k .$$

Note: Einstein's equivalence principle still holds if we add further terms containing spatial derivatives of g_i ie by replacing

$$g_i \rightarrow g_i - K_1 \triangle g_i \quad + \text{higher} \quad \text{order} \quad \text{terms}$$

where $\triangle$ is the Laplacian.

Self-gravitating fluid

Lagrangian formulation Generalise the one-particle phase space coordinates $A_i \in (x_i, p_i, y_i, q_i)$ to the continuum labelled by $\vec{\xi} \in R^3$ (comoving coordinates)

$$A_i(t) \rightarrow A_i(\vec{\xi}, t)$$

The Lagrangian for the selfgravitating fluid then becomes (after elimination of p_i and y_i)

$$L = -n_0 \int d^3\xi \left(\dot{q}_i(\vec{\xi}, t)\dot{x}_i(\vec{\xi}, t) + q_i(\vec{\xi}, t) g_i(\vec{x}(\vec{\xi}, t), t) \right) + L_\phi$$

$$L_\phi = -\frac{1}{8\pi G}\int d^3x\,(g_i(\vec{x},t))^2.$$

G - Newton's gravitational constant, n_0 is the (constant) particle density in $\vec{\xi}$-space.

Euler-Lagrange EOM

Varying q_i and x_i leads to the continuum generalisation of the previous equations for x_i and q_i.

Two possibilities to go further:

Either

Take $g_i = -\partial_i \phi$ and vary ϕ getting

$$\nabla \phi = 4\pi G \, \partial_k (n q_k) \quad \textbf{Poisson} \quad \textbf{equation}.$$

Note: The right hand side of this equation describes a dynamically generated active gravitational mass density which may be of either sign.

+ sign leading to attractive gravitation

- sign leading to repulsive gravitation.

This promotes the self-gravitating fluid to a possible candidate for the dark sector of the Universe.

These EOM are invariant w.r.t. arbitrary time-dependent translations (**gauge transformations**)

$$x_i \rightarrow x_i + a_i(t),$$

where ϕ transforms as

$$\phi(\vec{x}, t) \rightarrow \phi'(\vec{x}', t) = \phi(\vec{x}, t) - \ddot{a}_i(t) x_i$$

and q_i and n remain invariant.

Or

Varying g_i leads to a linear relation between q_i and g_i

$$g_i(\vec{x}, t) \;=\; -4\pi\, G\, n(\vec{x}, t)\, q_i(\vec{\xi}(\vec{x}, t)t)$$

with

$$n(\vec{x}, t) \;=\; n_0\, J^{-1} \quad \text{(particle density)}$$

and

$$J \;=\; \det\left(\frac{\partial x_i(\vec{\xi}, t)}{\partial \xi_k}\right) \quad \text{(Jacobian)}$$

Note that the expression for g_i can be rewritten as

$$q_i(\vec{\xi}, t) = -\frac{1}{4\pi G}\left(\frac{g_i}{n}\right)(\vec{x}(\vec{\xi}, t), t)$$

The last expression allows to **eliminate** from the EOM the **unphysical phase space variable** q_i in favour of the gravitational force g_i.
This removes the unphysical degrees of freedom. However, this expression for q_i (a particular solution of the Poisson equation) destroys the gauge symmetry *ie* it fixes the gauge.

Remark: The gauge symmetry can be restored by considering a curl-free velocity field

$$\vec{u} = \vec{\nabla} u$$

and treating u as the variable to be varied.

From the expression for q_i we get two coupled EOM for $\vec{x}(\vec{\xi}, t)$ and $\vec{g}(\vec{x}(\vec{\xi}, t), t)$

$$\frac{d^2}{dt^2}\, x_i(\vec{\xi}, t) \;=\; g_i(\vec{x}(\vec{\xi}, t)t)$$

(initial condition $\vec{x}(\vec{\xi}, 0) = \vec{\xi}$) and

$$\frac{d^2}{dt^2}\left(\frac{g_i}{n}\right)(\vec{x}(\vec{\xi}, t), t) \;=\; \left(\frac{1}{2n}\,\partial_i\, g_k^2\right)(\vec{x}(\vec{\xi}, t), t).$$

The last equation can be integrated once. To do this we note that

$$\theta_i \;=\; \frac{\partial \dot{x}_k}{\partial \xi_i}\,\frac{g_k}{n} \;-\; \frac{\partial x_k}{\partial \xi_i}\,\frac{d}{dt}\left(\frac{g_k}{n}\right)$$

is conserved (ie $\frac{d}{dt}\theta_i = 0$).

This conservation and the continutity equation

$$\frac{d}{dt} n + n \, \partial_k \, \dot{x}_k = 0$$

give the required integrated form of the equation for g_i

$$\frac{d}{dt} g_i = g_k \partial_i \, \dot{x}_k - g_i \partial_k \, \dot{x}_k - n \theta_k \, \frac{\partial \xi_k}{\partial x_i}.$$

Note that

$$Q = \int d^3\xi \, \theta_k(\vec{\xi}) \, \alpha_k(\vec{\xi})$$

generates infinitesimal relabeling transformations

$$\vec{\xi} \to \vec{\xi} + \vec{\alpha}(\vec{\xi}).$$

Eulerian formulation

Consider the map of independent variables

$$(\vec{\xi}, t) \rightarrow (\vec{x}, t) \qquad \text{with} \quad \vec{x} = \vec{x}(\vec{\xi}, t).$$

In addition map also

$$\frac{d}{dt} \rightarrow D_t = \frac{\partial}{\partial t} + u_k(\vec{x}, t)\partial_k \qquad \text{(convective derivative)}$$

with

$$\vec{u}(\vec{x}, t) = \frac{d}{dt}\vec{x}(\vec{\xi}, t)|_{\vec{\xi}=\vec{\xi}(\vec{x},t)}.$$

The EOM then become

$$D_t\, u_i \;=\; g_i$$

$$D_t\, g_i \;=\; g_k\, \partial_i u_k \;-\; g_i\, \partial_k u_k \;-\; n\theta_i$$

where

$$\theta_i(\vec{x}, t) \;=\; \frac{\partial \xi_k}{\partial x_i}\, \theta_k(\vec{\xi})|_{\vec{\xi}=\vec{\xi}(\vec{x},t)}$$

obeys

$$D_t\, \theta_i \;+\; \theta_k\, \partial_i u_k \;=\; 0$$

and n obeys the continuity equation

$$\partial_t n \;+\; \partial_k(n u_k) \;=\; 0.$$

Lagrangian All these equations are the Euler-Lagrange EOM of

$$L = -\frac{1}{4\pi G}\int d^3x \left(g_i(D_t u_i - \frac{1}{2}g_i) + \theta(\partial_t n + \partial_k(nu_k)) - n\alpha D_t\beta \right)$$

Here θ, α, β are the Clebsch parametrisation of the vector field θ_i

$$\theta_i = \partial_i\theta + \alpha\partial_i\beta.$$

Remarks: The EOM do not contain the gravitational constant G. The Lagrangian contains G only as a common factor. The Lagrangian does not split into free and interacting parts.Thus the fluid particles (darkons **do not exist as free particles** but are only elements of a self-gravitating fluid. This comes from the **reduction of the phase space** (expression of q_i in terms of g_i).

Symmetries

In addition to gauge and relabeling symmetry the model possesses unextended Galilei algebra enlarged by anisotropic dilational symmetry with $z = \frac{5}{3}$.

Comments: **Zero mass**

Given the generator of space translations

$$P_i = \int d^3x \, n \, p_i, \qquad \text{where}$$

$$np_i = -n \, D_t q_i = \frac{1}{4\pi G} \left(g_k \partial_i u_k - n\theta_i \right)$$

the generator of boosts becomes

$$K_i = P_i t - \frac{1}{4\pi g} \int d^3x \, g_i.$$

Then due to

$$[P_i,\, g_k(\vec{x},t)] \;=\; \partial_i\, g_k(\vec{x},t) \qquad \text{we} \quad \text{have}$$

$$[K_i,\, P_j] \;=\; 0$$

showing that the total mass of the darkon fluid is zero.

Anisotropic scaling

The model possesses a conserved dilation generator

$$D = Ht - \frac{1}{10\pi G}\int d^3x\, g_i u_i - \frac{3}{5}\int d^3x\, x_i n p_i.$$

Clearly

$$[P_i, D] = \frac{3}{5} P_i$$

and so $z = \frac{5}{3}$ *ie* space and time scale as

$$x_i^\star = \lambda x_i, \quad t^\star = \lambda^{\frac{5}{3}} t.$$

Remark: Anisotropic scaling emerges as a symmetry resulting from the minimal gravitational coupling. The addition of higher order derivatives terms would alter or destroy it.

Cosmological solutions

We consider the darkon fluid as a model for the dark sector of the Universe. Hence we look for solutions of the fluid EOM satisfying the cosmological principles (the universe being isotropic and homogeneous on large scales).
This gives

$$n(\vec{x}, t) = n(t), \quad u_i(\vec{x}, t) = x_i H(t).$$

$H(t)$, the **Hubble 'parameter'**, is given in terms of the **cosmic scale factor** $a(t)$ by

$$H = \frac{\dot{a}}{a},$$

and

$$g_i(\vec{x}, t) \;=\; -4\pi\, n(t)\, g(a(t))\, x_i$$

Note that the derivatives terms in the gravitational coupling would not contribute!.
Put all this into EOM and obtain:

1. From the continuity equation ($\dot{n} + 3\frac{\dot{a}}{a}n = 0$), after integration

$$n(t) = \frac{D}{\frac{4\pi}{3}a^3(t)}$$

where D=const > 0.

2. From the Euler eq.

$$-\ddot{a} = \frac{3D}{a^2}g(a)$$

ie a Friedmann-like equation

3. From the EOM for θ_i and g_i

$$\dot{g}(a(t)) = \frac{\beta}{a^2(t)}, \quad \beta = \text{const.}$$

The last two equations, after integration, give us a cubic equation for $g(a)$

$$g(g^2 + C_1) + C_0\left(1 - \frac{a_t}{a}\right) = 0,$$

with C_i integration constants and

$$a_t = \frac{2\beta^2}{DC_0}.$$

Next take $C_i > 0$ and use $a(t)$ as a measure of time (given that now we have $\dot{a}(t) > 0$)

Then

1. For $a < a_t$ we have $g(a) > 0$ and so $\ddot{a} < 0$ - *ie* deceleration phase of the early Universe.
2. For $a > a_t \rightarrow g(a) < 0$ and so $\ddot{a} > 0$ and we are in the accelaration phase of the late Universe.

Notes

- a_t defines the point of transition between the phases. All this is consistent with cosmological observations.
- We need $C_i > 0$ for these results.
- If $C_i = 0$ we reproduce the scale invariant solution valid at small t (then $a(t) \sim t^{\frac{3}{5}}$).

Comparison with other models

- No free parameters in the microscopic formulation of the model.
- The signs of C_i have to be explained by some physical arguments
- All other models contain either a new parameter (Λ in the standard ΛCDM), an unknown function $f(R)$ in modified gravity models, or a potential $V(\psi)$ for a scalar field ψ in some modified matter models).

Steady flow solutions

Question: Can a steady darkon fluid flow be a model for halos? *ie* provide a gravitational potential for the formation of stable structures (galaxies or clusters of galaxies)?

Comment We need to find steady solutions of our EOM leading to a potential ϕ which confines massive objects. We have not found such solutions yet. Only have some partial results. **Work in progress**.

Preliminary results

The problem has been reduced to finding 'relevant' solutions of the following set of coupled system on nonlinear PDEs (incompressible case)

$$\partial_i \, u_i \, = \, 0,$$

$$u_k \, \partial_k \, u_i \, = \, -\partial_i \, \phi,$$

$$u_k \, u_l \, \partial_k \, \partial_l \, \partial_i \, \phi \, = \, 0.$$

If $\theta_i = 0$ the last equation changes to

$$u_k \, \partial_k \, \partial_i \, \phi \, = \, \partial_k \, \phi \, \partial_i \, u_k.$$

Observations

- These equations are not compatible with each other in the case of a spherical symmetric potential $\phi = \phi(r)$.
- Halos are expected to have a triaxial ellipsoidal shapes.
- A simple ellipsoidal solution of the equations is given by

$$\phi \;=\; -c \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} \qquad \text{and} \quad u_i \;=\; \lambda_i \, x_i$$

with $\lambda_3 = -(\lambda_1 + \lambda_2)$ and $a_i^2 = \frac{c}{\lambda_i^2}$.

- **This potential is deconfining (wrong sign)**
- More work is required (perhaps one has to add derivative terms in the gravitational coupling)

Final remarks
Achievements

- Parameter free Lagrangian for a self-gravitating system of Galilean massless particles (darkon fluid)
- Dynamical generation of an active gravitational mass density of either sign.
- Explanation of the deceleration phase of the early Universe and of the acceleration phase of the late one.

Open problems

- Determination, from physical arguments, of unknown integrations constants (like C_i)
- Existence of halos?
- Embedding of the model into a larger one which possesses general relativistic limit (as $z = \frac{5}{3}$ our model cannot be a nonrelativistic limit of a relativistic one ($z = 1$)).
- ?

Thank you for your attention!