1 Conformal transformations in 2d

A. Conformal transformations of the coordinates leave the metric tensor invariant up to a scale:

$$g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x)$$

In two dimensions:

Concerning the change of metric tensor elements for a conformal transformation $x \rightarrow w(x)$:

$$g^{\mu\nu} \rightarrow \left( \partial w^\mu / \partial x^\alpha \right) \left( \partial w^\nu / \partial x^\beta \right) g^{\alpha\beta}$$

one gets conditions for the conformal transformation:

$$\left( \frac{\partial w^0}{\partial x^0} \right)^2 + \left( \frac{\partial w^1}{\partial x^1} \right)^2 = \left( \frac{\partial w^1}{\partial x^0} \right)^2 + \left( \frac{\partial w^0}{\partial x^1} \right)^2$$

and

$$\left( \frac{\partial w^0}{\partial x^1} \right) \left( \frac{\partial w^1}{\partial x^0} \right) + \left( \frac{\partial w^0}{\partial x^0} \right) \left( \frac{\partial w^1}{\partial x^1} \right) = 0$$

These are the Cauchy-Riemann equations for a holomorphic function:

$$\frac{\partial w^0}{\partial x^0} = -\frac{\partial w^1}{\partial x^1}, \quad \frac{\partial w^1}{\partial x^0} = \frac{\partial w^0}{\partial x^1}$$

or for an anti-holomorphic function:

$$\frac{\partial w^0}{\partial x^0} = \frac{\partial w^1}{\partial x^1}, \quad \frac{\partial w^1}{\partial x^0} = -\frac{\partial w^0}{\partial x^1}$$

Writing above conditions in terms of complex variables

$$z = x^0 + ix^1, \quad \bar{z} = x^0 - ix^1,$$

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_0 - i\partial_1), \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1),$$

$$f = w^0 + iw^1, \quad \bar{f} = w^0 - iw^1$$

one arrives at the explicit formulae:

$$\bar{\partial} f(z) = 0 \quad \text{or} \quad \partial \bar{f}(\bar{z}) = 0$$

Conclusion: In two dimensions local conformal transformations are realized by holomorphic functions $f(z)$.

Conformal transformations that can be defined globally on the complex plane have the form:

$$f(z) = \frac{az + b}{cz + d} \quad \text{with} \quad ad - bc = 1$$
They form $SL(2,\mathbb{C})$ group.

B. Conformal map from a cylinder to the complex plane

$$\xi \rightarrow z = e^\xi = e^{\tau + i\sigma}$$

- The Euclidean time infinities are mapped to zero and the point in infinity:
  $$\tau = \pm \infty \rightarrow z = 0, \infty,$$
- equal time $\rightarrow |z| = const$,
- time reversal: $\tau \rightarrow -\tau : z \rightarrow \frac{1}{\bar{z}^*}$,
- time translations $\tau \rightarrow \tau + a : z \rightarrow e^a z$ - dilatations

2 The free massless scalar

2.1 Action

$$S = \frac{1}{4\pi} \int dz d\bar{z} \partial \Phi(z, \bar{z}) \bar{\partial} \Phi(z, \bar{z})$$

- propagator (2-point function)
  $$\partial \bar{\partial} \langle \Phi(z, \bar{z}) \Phi(w, \bar{w}) \rangle = -4\pi \delta^{(2)}(z - w)$$

- solution of the equation:
  $$\langle \Phi(z, \bar{z}) \Phi(w, \bar{w}) \rangle = -\ln |z - w|^2 \tag{1}$$

2-point function of derivatives:

$$\langle \partial_z \Phi(z, \bar{z}) \partial_w \Phi(w, \bar{w}) \rangle = -\frac{1}{(z - w)^2}$$

$$\langle \bar{\partial}_z \Phi(z, \bar{z}) \bar{\partial}_w \Phi(w, \bar{w}) \rangle = -\frac{1}{(\bar{z} - \bar{w})^2}$$

From classical equation of motion $\partial \bar{\partial} \Phi = 0$ it follows that the derivatives of scalar field are holomorphic and anti-holomorphic fields. We use notation:

$$\partial \Phi(z, \bar{z}) \equiv \partial \phi(z), \quad \bar{\partial} \Phi(z, \bar{z}) \equiv \bar{\partial} \phi(\bar{z})$$
• energy-momentum tensor is defined in terms of the action:

\[ \delta S = \int d^d x \ T^{\mu \nu} \partial_{\mu} \epsilon_{\nu} \]

From the Noether’s theorem it has the form:

\[ T^{\mu \nu} = g^{\mu \nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \Phi)} \partial^{\nu} \Phi \]

Using

\[ g^{\mu \nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad g_{\mu \nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \partial \bar{z} = \partial_{\bar{z}} \]

\[ T^{z \bar{z}} = 2 \frac{\partial \mathcal{L}}{\partial \partial_{\bar{z}} \Phi} \partial_{\Phi} = \frac{1}{2\pi} \partial_{\Phi} \partial_{\bar{z}} \Phi \]
\[ T^{zz} = 2 \frac{\partial \mathcal{L}}{\partial \partial_{z} \Phi} \partial_{\bar{z}} = \frac{1}{2\pi} \partial_{\bar{z}} \partial_{\Phi} \Phi \]
\[ T^{z \bar{z}} = 2 \frac{\partial \mathcal{L}}{\partial \partial_{\bar{z}} \Phi} \partial_{\Phi} - 2\mathcal{L} = 0, \quad T^{\bar{z} z} = 0 \]

The two non-vanishing components are the holomorphic and the anti-holomorphic one:

\[ T(z) = -4\pi T_{z z} = -\pi T^{z \bar{z}}, \quad \bar{T}(\bar{z}) = -4\pi T_{z \bar{z}} = -\pi T^{\bar{z} z} \]

This suggest the definition of holomorphic field (anti-holomorphic \( \bar{T}(\bar{z}) \) analogously)

\[ T(z) = -\frac{1}{2} : \partial \phi(z) \partial \phi(z) := -\frac{1}{2} \lim_{w \to z} [\partial \phi(z) \partial \phi(w) - \langle \partial \phi(z) \partial \phi(w) \rangle] \quad (2) \]

where : : stands for normal ordering

• Operator Product Expansions (OPEs) with \( T(z) \)

Using definition (2) and Wick theorem we can compute

\[ T(z) \partial \phi(w) = -\frac{1}{2} : \partial \phi(z) \partial \phi(z) : \phi(w) = \frac{\partial \phi(w)}{(z - w)^2} + \frac{\partial^2 \phi(w)}{(z - w)} + \text{reg.} \]

\[ T(z) : e^{i\alpha \Phi(w, \bar{w})} : = \frac{\alpha^2/2}{(z - w)^2} : e^{i\alpha \Phi(w, \bar{w})} : + \frac{1}{(z - w)} \partial_w : e^{i\alpha \Phi(w, \bar{w})} : + \text{reg.} \]
2.2 The energy-momentum tensor as the generator of conformal transformations

2.2.1 Conditions for the energy-momentum tensor

According to Noether’s theorem to every continuous symmetry of a field theory corresponds a conserved current and thus the conserved charge. The conserved current associated with conformal symmetry is given by energy-momentum tensor:

\[ j^\mu = T^{\mu\nu} \epsilon_\nu, \quad \partial_\mu j^\mu = 0 \]

- special case of translations \( \epsilon_\mu = \text{const} \) implies continuity equation:
  \[ 0 = \partial_\mu (T^{\mu\nu} \epsilon_\nu) = (\partial_\mu T^{\mu\nu}) \epsilon_\nu \quad \Rightarrow \quad \partial_\mu T^{\mu\nu} = 0 \]

- case of \( \epsilon_\mu(x) \)
  \[ 0 = \partial_\mu (T^{\mu\nu} \epsilon_\nu) = (\partial_\mu T^{\mu\nu}) \epsilon_\nu + T^{\mu\nu} \partial_\mu \epsilon_\nu = \frac{1}{2} T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = -\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} \]

where we use relation for \( \delta g_{\mu\nu} \):

\[ \delta g_{\mu\nu} = g'_{\mu\nu} - g_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'\mu} \frac{\partial x^\beta}{\partial x'\nu} g_{\alpha\beta} - g_{\mu\nu} = (\delta_\mu^\alpha - \partial_\mu \epsilon_\alpha)(\delta_\nu^\beta - \partial_\nu \epsilon_\beta) g_{\alpha\beta} - g_{\mu\nu} = -(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \]

For conformal transformations:

\[ \delta g_{\mu\nu} = \lambda(x) g_{\mu\nu} \]

the conservation law implies traceless of energy-momentum tensor:

\[ \partial_\mu j^\mu = 0 \quad \Rightarrow \quad \frac{1}{2} T^{\mu\nu} \lambda(x) g_{\mu\nu} = \frac{1}{2} T^{\mu\nu} \lambda(x) = 0 \]

- in complex coordinates the traceless of energy-momentum tensor together with the continuity equation give conditions for components of tensor:
  \[ \bar{\partial} T_{zz} = 0, \quad \partial T_{\bar{z}z} = 0, \quad T_{z\bar{z}} = T_{\bar{z}z} = 0 \]

Conclusion: Non-vanishing components of the energy-momentum tensor are holomorphic and anti-holomorphic functions:

\[ T_{zz} = T(z), \quad T_{z\bar{z}} \equiv \bar{T}(\bar{z}) \]
2.2.2 Conserved charge

Since the current \( j^\mu = T^{\mu\nu} e_\nu \) associated with the conformal symmetry is preserved there exists a conserved charge:

\[
Q = \int dx^1 j_0 \quad \text{at} \quad x^0 = \text{const}
\]

It is the generator of symmetry transformations for an operator \( A \):

\[
\delta A = [Q, A],
\]

where we have equal-times commutator. In complex coordinates \( x^0 = \text{const} \to |z| = \text{const} \), so

\[
\int dx^1 \to \oint dz
\]

with convention that integral contours are clockwise. Then the conserved charge

\[
Q = \oint \frac{1}{2\pi i} \left( dz T(z) \epsilon(z) + d\bar{z} \bar{T}(\bar{z}) \bar{\epsilon}(\bar{z}) \right)
\]

Thus the variation of a field \( \delta \varphi = [Q, \varphi] \) is given by commutators:

\[
\delta_{\epsilon, \bar{\epsilon}} \varphi(w, \bar{w}) = \oint \frac{dz}{2\pi i} \left[ T(z) \epsilon(z), \varphi(w, \bar{w}) \right] + \oint \frac{d\bar{z}}{2\pi i} \left[ \bar{T}(\bar{z}) \bar{\epsilon}(\bar{z}), \varphi(w, \bar{w}) \right]
\]

- The variation of the derivative of free scalar field (to compute we need OPE \( T(z) \partial \phi(w) \)):

\[
\delta_{\epsilon} \left( \partial \phi \right)(w) = \oint \frac{dz}{2\pi i} T(z) \epsilon(z) \partial \phi(w) = \oint \frac{dz}{2\pi i} \epsilon(z) \left\{ \frac{\partial \phi(w)}{(z - w)^2} + \frac{\partial^2 \phi(w)}{(z - w)} \right\}
\]

We can compare this with the transformation rule for scalar field:

\[
w \to w + \epsilon(w), \quad \Phi(w, \bar{w}) \to \Phi(w + \epsilon, \bar{w} + \bar{\epsilon}) = \Phi(w, \bar{w}) + \epsilon(w) \partial \Phi(w, \bar{w}) + \bar{\epsilon} \bar{\partial} \Phi(w, \bar{w})
\]

\[
\partial \phi(w) \to \partial \phi(w) + \epsilon(w) \partial \phi(w) + \bar{\epsilon}(w) \bar{\partial} \phi(w)
\]

- Appendix: The equal time commutators and radial ordering

We consider an operator in the form of integral with contour enclosing zero \( A = \oint \frac{dz}{2\pi i} a(z) \). The commutator:

\[
[A, b(w)] = \lim_{|z| \to |w|} \left( \oint_{|z| > |w|} \frac{dz}{2\pi i} a(z) b(w) - \oint_{|z| < |w|} \frac{dz}{2\pi i} b(w) a(z) \right) = \oint \frac{dz}{2\pi i} \mathcal{R}(a(z) b(w))
\]

where radial ordering:

\[
\mathcal{R}(a(z) b(w)) = \begin{cases} 
 a(z) b(w) & \text{if } |z| > |w|, \\
 b(w) a(z) & \text{if } |z| < |w| 
\end{cases}
\]
2.2.3 Conformal transformations of energy-momentum tensor

- infinitesimal variation
  \[ \delta_{\epsilon} T(w) = \oint_w \frac{dz}{2\pi i} T(z) \epsilon(z) T(w) \]  

We need to know the OPE:

\[ T(z)T(w) = \frac{1}{4} : \partial \phi(z) \partial \phi(z) : \partial \phi(w) \partial \phi(w) : = \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \]

The variation of \( T(z) \) has the form:

\[ \delta_{\epsilon} T(w) = \oint_w \frac{dz}{2\pi i} \left[ \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots \right] = \frac{1}{12} \partial^2 \epsilon(w) + 2T(w)\partial \epsilon(w) + \epsilon(w)\partial T(z) \]

- for a finite transformation \( z \to w(z) \)

We write the tensor in the form:

\[ T(z) = -\frac{1}{2} \lim_{\delta \to 0} \left( \partial \phi(z + \frac{\delta}{2}) \partial \phi(z - \frac{\delta}{2}) + \frac{1}{\delta^2} \right) \]

The derivative \( \partial \phi \) transforms as

\[ \partial_z \phi(z) = \left( \frac{dw}{dz} \right) \left( \frac{\partial \phi(w)}{\partial w} \right) = w^{(1)} \partial_w \phi'(w) \]

Thus the tensor transforms as

\[
T(z) = -\frac{1}{2} \lim_{\delta \to 0} \left[ w^{(1)}(z + \frac{\delta}{2}) w^{(1)}(z - \frac{\delta}{2}) \partial_w \phi'(w(z + \frac{\delta}{2})) \partial_w \phi'(w(z - \frac{\delta}{2})) + \frac{1}{\delta^2} \right]
\]

\[
= -\frac{1}{2} \lim_{\delta \to 0} \left[ w^{(1)}(z + \frac{\delta}{2}) w^{(1)}(z - \frac{\delta}{2}) \left( \partial_w \phi'(w) \partial_w \phi'(w) : - \frac{1}{(w(z + \frac{\delta}{2}) - w(z - \frac{\delta}{2}))^2} + \frac{1}{\delta^2} \right) \right]
\]

\[
= (w^{(1)}(z))^2 T'(w) + \frac{1}{2} \lim_{\delta \to 0} \left[ \frac{w^{(1)}(z + \frac{\delta}{2}) w^{(1)}(z - \frac{\delta}{2})}{(w(z + \frac{\delta}{2}) - w(z - \frac{\delta}{2}))^2} - \frac{1}{\delta^2} \right]
\]

\[
= (w^{(1)}(z))^2 T'(w) + \frac{1}{12} \left[ \frac{w^{(3)}(z)}{w^{(1)}(z)} - \frac{3}{2} \left( \frac{w^{(2)}(z)}{w^{(1)}(z)} \right)^2 \right] .
\]

and finally

\[ T'(w) = \left( \frac{dw}{dz} \right)^{-2} \left[ T(z) - \frac{1}{12} S(w, z) \right] \]
where the last term is the Schwarzian derivative

\[ S(w, z) = \frac{1}{(\partial_z w)^2} \left( (\partial_z w)(\partial_z^2 w) - \frac{3}{2}(\partial_z^2 w)^2 \right) \]

It vanishes for global conformal transformations and satisfies the following relation

\[ S(u, z) = S(w, z) + \left( \frac{dw}{dz} \right)^2 S(u, w), \]

which assure that the result of two successive transformations \( z \rightarrow w \rightarrow u \) coincides with the result of the single transformation \( z \rightarrow u \):

\[
T''(u) = \left( \frac{du}{dw} \right)^{-2} \left[ T'(w) - \frac{1}{12} S(u, w) \right] = \left( \frac{du}{dz} \right)^{-2} \left[ \left( \frac{dw}{dz} \right)^{-2} \left[ T(z) - \frac{1}{12} S(w, z) \right] - \frac{1}{12} S(u, w) \right] = \left( \frac{du}{dz} \right)^{-2} \left[ T(z) - \frac{1}{12} S(u, z) \right]
\]

It can be showed that the Schwarzian derivative is the only possible addition to the tensor transformation law that has these two properties. In general the Schwarzian derivative can be multiplied by an arbitrary factor:

\[ T'(w) = \left( \frac{dw}{dz} \right)^{-2} \left[ T(z) - \frac{c}{12} S(w, z) \right] \]

The parameter \( c \) is called the \textbf{central charge}. For the free scalar theory \( c = 1 \). The infinitesimal version of the transformation law above reads:

\[
T'(z + \epsilon) = T'(z) + \epsilon(z) \partial T(z) = (1 - 2\partial \epsilon(z)) \left( T(z) - \frac{c}{12} \partial^3 \epsilon(z) \right),
\]

what coincides with (4) for the following OPE of two tensors:

\[ T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \quad (5) \]

\subsection*{2.2.4 Modes of energy-momentum tensor}

We define modes of energy-momentum tensor:

\[
T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \quad L_n = \frac{1}{2\pi i} \oint dz \, z^{n+1} T(z),
\]

\[
\bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n, \quad \bar{L}_n = \frac{1}{2\pi i} \oint d\bar{z} \, \bar{z}^{n+1} \bar{T}(\bar{z}). \quad (6)
\]
Writing a general infinitesimal conformal transformation in the form
\[ z' = z + \epsilon(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1} \]
we get the relation between conserved charge \( Q_\epsilon \) and modes \( L_n \):
\[
Q_\epsilon = \oint \frac{dz}{2\pi i} \epsilon(z) T(z) = \oint \frac{dz}{2\pi i} \sum_n \epsilon_n z^{n+1} T(z) = \sum_n \epsilon_n L_n
\]
The modes \( L_n \) are generators of infinitesimal conformal transformations. They satisfy the following commutation relations:
\[
[L_n, L_m] = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+1} = \oint \frac{dw}{2\pi i} w^{n+1} \oint \frac{dz}{2\pi i} z^{m+1} T(z) T(w)
\]
\[
= \oint \frac{dw}{2\pi i} w^{n+1} \oint \frac{dz}{2\pi i} z^{m+1} \left( \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \right)
\]
\[
= (n - m) L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0}
\]
This is the Virasoro algebra with central charge \( c \). The modes \( L_n \) are called Virasoro generators. Analogously for the anti-holomorphic tensor \( T(\bar{z}) \) we have:
\[
[L_n, \bar{L}_m] = (n - m) \bar{L}_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0}
\]
\[
[L_n, \bar{L}_m] = 0
\]

### 2.3 Canonical quantization

We consider the field defined on a cylinder of circumference \( L \): \( \Phi(x + L, t) \equiv \Phi(x, t) \). The Fourier mode expansion:
\[
\Phi(x, t) = \sum_n e^{2\pi inx/L} \varphi_n(t)
\]
\[
\varphi_n(t) = \frac{1}{L} \int dx \ e^{-2\pi inx/L} \Phi(x, t)
\]
The free field Lagrangian
\[
\frac{1}{8\pi} \int dx \ [\left( \partial_t \Phi \right)^2 - \left( \partial_x \Phi \right)^2]
\]
in terms of Fourier modes:
\[
\frac{L}{8\pi} \sum_n \left[ \dot{\varphi}_n \dot{\varphi}_{-n} - \left( \frac{2\pi n}{L} \right)^2 \varphi_n \varphi_{-n} \right]
\]
The momentum conjugate to \( \varphi_n \):
\[
\pi_n = \frac{\partial L}{\partial \dot{\varphi}_n} = \frac{L}{4\pi} \varphi_{-n}
\]
The Hamiltonian has the form
\[
\frac{2\pi}{L} \sum_n \left[ \pi_n \pi_{-n} + \left( \frac{n}{2} \right)^2 \varphi_n \varphi_{-n} \right]
\]
In canonical quantization we replace \( \varphi_n, \pi_n \) by operators and impose canonical commutation
rules:
\[
[\varphi_n, \pi_m] = i \delta_{mn}, \quad [\varphi_n, \varphi_m] = [\pi_n, \pi_m] = 0
\]
Define the raising and lowering operators in terms of \( \tilde{a}_n, \tilde{a}^\dagger_n \) \((\varphi_0^\dagger = \varphi_{-n}, \pi_0^\dagger = \pi_{-n})\):
\[
\tilde{a}_n = \frac{1}{\sqrt{|n|}} \left( \frac{|n|}{2} \varphi_n + i \pi_{-n} \right)
\]
so that
\[
\begin{aligned}
a_n &= \begin{cases}
-i \sqrt{n} \tilde{a}_n & \text{if } n > 0, \\
 i \sqrt{-n} \tilde{a}_n^\dagger & \text{if } n < 0
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
\tilde{a}_n &= \begin{cases}
-i \sqrt{n} \tilde{a}_{-n} & \text{if } n > 0, \\
 i \sqrt{-n} \tilde{a}_{-n}^\dagger & \text{if } n < 0
\end{cases}
\end{aligned}
\]
The zero mode will be treated separately. The commutation relation satisfied by raising and
lowering (creation and annihilation) operators:
\[
[a_n, a_m] = n \delta_{n+m}, \quad [a_n, \bar{a}_m] = 0 \quad [\bar{a}_n, \bar{a}_m] = n \delta_{n+m}
\]
The Hamiltonian written in terms of above operators takes the form:
\[
H = \frac{2\pi}{L} \pi_0^2 + \frac{2\pi}{L} \sum_{n \neq 0} (a_{-n} a_n + \bar{a}_{-n} \bar{a}_n)
\]
The mode expansion of the field in arbitrary time in terms of creation and annihilation
operators:
\[
\Phi(x, t) = \varphi_0 + 4 \pi \frac{\pi_0 t}{L} + i \sum_{n \neq 0} \frac{1}{n} (a_n e^{2\pi i n(x-t)/L} - \bar{a}_{-n} e^{2\pi i n(x+t)/L})
\]
The evolution of operators \( \varphi_0, a_n, a_{-n} \) follows from the Hamiltonian:
\[
\dot{a}_n = i[H, a_n] = -i \frac{2\pi}{L} n a_n
\]
\[
a_n(t) = a_n(0) e^{-2\pi i n t / L}, \quad \bar{a}_n(t) = \bar{a}_n(0) e^{-2\pi i n t / L}
\]
\[
\varphi_0(t) = \varphi_0 + 4 \pi \frac{\pi_0 t}{L},
\]
The mode expansion of the field in arbitrary time in terms of creation and annihilation
operators:
Now we go to Euclidean space-time \((t \rightarrow -i\tau)\) and transform to complex plane:

\[z = e^{2\pi(\tau - ix)/L}, \quad \bar{z} = e^{2\pi(\tau + ix)/L}\]

The free scalar field takes the form:

\[\Phi(z, \bar{z}) = \varphi_0 - i\pi_0 \ln(z\bar{z}) + i\sum_{n \neq 0} \frac{1}{n}(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}) \quad (7)\]

The derivative \(\partial_z \Phi(z, \bar{z})\) is the holomorphic field with the following mode expansion:

\[i\partial\phi(z) = \frac{\pi_0}{z} + \sum_{n \neq 0} a_n z^{-n-1} \quad (8)\]

Introducing the operators

\[a_0 = \bar{a}_0 = \pi_0\]

we can include the zero mode in the sum:

\[i\partial\phi(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \quad (8)\]

- **Fock space**

  The Fock space is built on the 1-parameter family of vacua \(|\alpha\rangle\) - the eigenstate of \(\pi_0\) such that:

  \[a_n |\alpha\rangle = \bar{a}_n |\alpha\rangle = 0 \quad (n > 0), \quad a_0 |\alpha\rangle = \bar{a}_0 |\alpha\rangle = \alpha |\alpha\rangle\]

The modes \(a_n, \bar{a}_n\) are annihilation operators for \(n > 0\) and creation operators for \(n < 0\). The elements of Fock space \(\mathcal{F}_\alpha\) are obtained by the action of creation operators on the vacuum state \(|\alpha\rangle\):

\[a_{n_1}^{m_1} a_{n_2}^{m_2} ... \bar{a}_{n_1}^{m_1} \bar{a}_{n_2}^{m_2} ... |\alpha\rangle \quad (n_i, m_j \geq 0) \quad (9)\]

The vacuum states with different eigenvalues \(\alpha\) can be obtained from the one with \(\alpha = 0\) by the action of so called vertex operator \(V(z, \bar{z}) = e^{i\alpha \Phi(z, \bar{z})}\):

\[|\alpha\rangle = \lim_{z, \bar{z} \to 0} e^{i\alpha \Phi(z, \bar{z})} |0\rangle\]

Indeed, the state is the eigenstate of \(\pi_0\) to eigenvalue \(\alpha\) and it is annihilated by annihilation operators:
1. \[
\pi_0 e^{i\alpha \Phi(z,\bar{z})} |0\rangle = \alpha e^{i\alpha \Phi(z,\bar{z})} |0\rangle
\]

In order to see this we can compute commutator

\[
[\pi_0, e^{i\alpha \Phi(z,\bar{z})}] = e^{i\alpha \Phi(z,\bar{z})} [\pi_0, i\alpha \Phi(z,\bar{z})] = i\alpha e^{i\alpha \Phi(z,\bar{z})} [\pi_0, \varphi_0] = \alpha e^{i\alpha \Phi(z,\bar{z})}
\]

where we have used relations

\[
[B, e^A] = e^A [B, A], \quad [\pi_0, \varphi_0] = -i.
\]

2. \(a_n |\alpha\rangle = 0\) for \(n > 0\):

\[
[a_n, e^{i\alpha \Phi(z,\bar{z})}] = e^{i\alpha \Phi(z,\bar{z})} [a_n, i\alpha \Phi(z,\bar{z})] = i\alpha e^{i\alpha \Phi(z,\bar{z})} [a_n, \left( \frac{i}{-n} \right) a_{-n} z^n]
\]

\[
= \frac{1}{n} \alpha z^n e^{i\alpha \Phi(z,\bar{z})} [a_n, a_{-n}] = -\alpha z^n \alpha e^{i\alpha \Phi(z,\bar{z})}
\]

Acting on vacuum state in the limit \(z, \bar{z} \to 0\):

\[
\lim_{z,\bar{z} \to 0} [a_n, e^{i\alpha \Phi(z,\bar{z})}] |0\rangle = 0
\]

- modes of energy-momentum tensor and \(a_n\) operators

\[
T(z) = -\frac{1}{2} : \partial \phi(z) \partial \phi(z) : = \frac{1}{2} \sum_{n,m \in \mathbb{Z}} z^{-n-m-2} : a_n a_m :
\]

Relation between modes of energy-momentum tensor \(L_n\) and \(a_n\):

\[
T(z) = \sum_p z^{-p-2} L_p \quad \Rightarrow \quad L_p = \frac{1}{2} \sum_n a_{p-n} a_n \quad (n \neq 0),
\]

The \(L_0\) generator:

\[
L_0 = \sum_{n>0} a_{-n} a_n + \frac{1}{2} a_0^2.
\]

Without normal ordering in definition of \(T(z)\) there would be an additional infinite term coming from commutator \([a_n, a_{-n}] = n\). The anti-holomorphic modes:

\[
L_p = \frac{1}{2} \sum_n \bar{a}_{p-n} \bar{a}_n \quad (n \neq 0), \quad L_0 = \sum_{n>0} \bar{a}_{-n} \bar{a}_n + \frac{1}{2} \bar{a}_0^2
\]

One can check that the Hamiltonian can be written as

\[
H = \frac{2\pi}{L} (L_0 + \bar{L}_0)
\]
The Fock vacuum is the eigenstate of $L_0$ to eigenvalue $\frac{\alpha^2}{2}$:

$$L_0 |\alpha\rangle = \left( \sum_{n>0} a_{-n} a_n + \frac{1}{2} a_0^2 \right) |\alpha\rangle = \frac{\alpha^2}{2} |\alpha\rangle$$

An arbitrary state of the form $|\alpha\rangle$ is the eigenstate of $L_0, \bar{L}_0$:

$$L_0 a_{-n_1} a_{-n_2} \ldots a_{-n_m} |\alpha\rangle = \left( \frac{\alpha^2}{2} + \sum_{j} j n_j \right) |\alpha\rangle$$

$$\bar{L}_0 a_{-n_1} a_{-n_2} \ldots a_{-n_m} |\alpha\rangle = \left( \frac{\alpha^2}{2} + \sum_{j} j m_j \right) |\alpha\rangle$$

what can be deduced from the following commutators:

$$[L_0, a_{-k}] = ka_{-k}, \quad [L_0, \bar{a}_{-k}] = 0.$$

• The Fock space builded on the vacuum state $|\alpha\rangle$ can be written as a tensor products of left and right Fock spaces generated by left $a_n$ and right $\bar{a}_n$. The space of all states is a sum (or integral) over all possible parameters $\alpha$:

$$\mathcal{H} = \int d\alpha \mathcal{F}_\alpha \otimes \bar{\mathcal{F}}_\alpha$$

### 2.4 Normal ordered exponents

$$F^\alpha (z, \bar{z}) := e^{i\alpha \varphi_0 + \alpha \varphi_<(z) + \alpha \varphi_<(\bar{z})} (z \bar{z})^{\alpha \pi_0} e^{-\alpha \varphi_>(z) - \alpha \varphi_>(\bar{z})}$$

where

$$\varphi_<(z) = \sum_{n>0} \frac{1}{n} a_{-n} z^n, \quad \varphi_>(z) = \sum_{n>0} \frac{1}{n} a_n z^{-n},$$

### 2.4.1 Exchange relations

We will need the commutator:

$$[\varphi_>(z), \varphi_<(w)] = \left( \sum_{n>0} \frac{1}{n} a_{-n} z^{-n}, \sum_{m>0} \frac{1}{m} a_{-m} w^m \right] = \sum_{n,m>0} \frac{1}{nm} z^{-n} w^m [a_n, a_{-m}]$$

$$= \sum_{n>0} \frac{1}{n} (w/z)^n = -\ln \left( 1 - \frac{w}{z} \right)$$

For the exponents we have

$$e^{-\alpha \varphi_>(z)} e^{\beta \varphi_<(w)} = e^{\beta \varphi_<(w)} e^{-\alpha \varphi_>(z)} e^{-\alpha \beta [\varphi_>(z), \varphi_<(w)]} = \left( 1 - \frac{w}{z} \right)^{\alpha \beta} e^{\beta \varphi_<(w)} e^{-\alpha \varphi_>(z)}$$ (10)
where we use the Hausdorff formula

\[ e^A e^B = e^B e^A e^{[A,B]} \]

Using this relation we can compute also

\[ (z \bar{z})^{\alpha \pi_0} e^{i \beta \rho_0} = e^{i \beta \rho_0} (z \bar{z})^{\alpha \pi_0} (z \bar{z})^{i \alpha \beta (\alpha \pi_0, \rho_0)} = e^{i \beta \rho_0} (z \bar{z})^{\alpha \pi_0} (z \bar{z})^{\alpha \beta} \tag{11} \]

### 2.4.2 OPE of two exponents

We can compute the OPE of two ordered exponents by ordering their product:

\[
E^\alpha(z, \bar{z}) \ E^\beta(w, \bar{w}) = e^{i \alpha \phi_0 + \alpha \phi <(z) + \alpha \phi <(\bar{z})} (z \bar{z})^{\alpha \pi_0} e^{-\alpha \phi >(z) - \alpha \phi >(\bar{z})} \\
\times e^{i \beta \phi_0 + \beta \phi <(w) + \beta \phi <(\bar{w})} (w \bar{w})^{\beta \pi_0} e^{-\beta \phi >(w) - \beta \phi >(\bar{w})} \\
= \left(1 - \frac{w}{z}\right)^{\alpha \beta} \left(1 - \frac{\bar{w}}{\bar{z}}\right)^{\alpha \beta} e^{i \alpha \phi_0 + \alpha \phi <(z) + \alpha \phi <(\bar{z})} (e^{\beta \phi <(w)} + e^{\beta \phi <(\bar{w})}) \\
\times (z \bar{z})^{\alpha \pi_0} e^{i \beta \rho_0} (e^{-\alpha \phi >(z) - \alpha \phi >(\bar{z})}) (w \bar{w})^{\beta \pi_0} e^{-\beta \phi >(w) - \beta \phi >(\bar{w})} \\
= (z - w)^{\alpha \beta} (z - \bar{w})^{\alpha \beta} e^{(\alpha + \beta) \rho_0} e^{\alpha \phi <(z) + \alpha \phi <(\bar{z})} e^{\beta \phi <(w) + \beta \phi <(\bar{w})} \\
\times (z \bar{z})^{\alpha \pi_0} (w \bar{w})^{\beta \pi_0} e^{-\alpha \phi >(z) - \alpha \phi >(\bar{z}) - \beta \phi >(w) - \beta \phi >(\bar{w})} \\
= (z - w)^{\alpha \beta} (z - \bar{w})^{\alpha \beta} E^{\alpha + \beta}(w, \bar{w}) + \ldots
\]

where we used first the relations for exchanging exponents \[10\] and then \[11\].

One can also compute the OPE of two exponents using Wick theorem:

\[
E^\alpha(z, \bar{z}) E^\beta(w, \bar{w}) = : e^{i \alpha \Phi(z, \bar{z})} : e^{i \alpha \Phi(w, \bar{w})} := \sum_{n=0}^{\infty} \frac{(i \alpha \Phi(z, \bar{z}))^n}{n!} : \sum_{m=0}^{\infty} \frac{(i \alpha \Phi(w, \bar{w}))^m}{m!} : \\
= \exp (-\alpha \beta < \Phi(z, \bar{z}) \Phi(w, \bar{w})>) : e^{i \alpha \Phi(z, \bar{z})} e^{i \alpha \Phi(w, \bar{w})} : \\
= (z - w)^{\alpha \beta} : e^{i \alpha \Phi(z, \bar{z})} e^{i \alpha \Phi(w, \bar{w})} :
\]

### 2.4.3 2-point function

\[
\langle E^\alpha(z, \bar{z}) E^\beta(w, \bar{w}) \rangle = (z - w)^{\alpha \beta} (\bar{z} - \bar{w})^{\alpha \beta} \left\{ e^{i (\alpha + \beta) \rho_0} e^{\alpha \phi <(z) + \alpha \phi <(\bar{z})} e^{\beta \phi <(w) + \beta \phi <(\bar{w})} \right\} \\
\times (z \bar{z})^{\alpha \pi_0} (w \bar{w})^{\beta \pi_0} e^{-\alpha \phi >(z) - \alpha \phi >(\bar{z}) - \beta \phi >(w) - \beta \phi >(\bar{w})} \\
= (z - w)^{\alpha \beta} (z - \bar{w})^{\alpha \beta} \delta_{\alpha, -\beta} = (z - w)^{-\alpha^2} (\bar{z} - \bar{w})^{-\alpha^2} \delta_{\alpha, -\beta}
\]
The condition on momentum eigenvalues

\[ \alpha + \beta = 0 \]

follows from the fact that operator \( e^{i\mu \varphi_0} \) shifts the momentum of the \( \pi_0 \) eigenstate:

\[ \pi_0 e^{i\mu \varphi_0} |\alpha\rangle = e^{i\mu \varphi_0} [\pi_0, i\mu \varphi_0] |\alpha\rangle + e^{i\mu \varphi_0} \pi_0 |\alpha\rangle = (\mu + \alpha) e^{i\mu \varphi_0} |\alpha\rangle \]

### 2.4.4 3-point function

\[
\langle E^{\alpha_3}(z_3, \bar{z}_3) E^{\alpha_2}(z_2, \bar{z}_2) E^{\alpha_1}(z_1, \bar{z}_1) \rangle
\]

\[
= (z_2 - z_1)^{\alpha_2 \alpha_1} (\bar{z}_2 - \bar{z}_1)^{\alpha_2 \alpha_1} \left( E^{\alpha_3}(z_3, \bar{z}_3) e^{ \delta(\alpha_1 + \alpha_2) \varphi_0} e^{ \alpha_2 \varphi < (z_2) + \alpha_2 \varphi < (\bar{z}_2) } e^{ \alpha_1 \varphi < (z_1) + \alpha_1 \varphi < (\bar{z}_1) } \right) \\
\times (z_2 - z_1)^{\alpha_2 \alpha_1} (\bar{z}_2 - \bar{z}_1)^{\alpha_2 \alpha_1} \left( e^{ \delta(\alpha_1 + \alpha_2) \varphi_0} e^{ \alpha_3 \varphi < (z_3) + \alpha_3 \varphi < (\bar{z}_3) } e^{ \alpha_2 \varphi < (z_2) + \alpha_2 \varphi < (\bar{z}_2) } e^{ \alpha_1 \varphi < (z_1) + \alpha_1 \varphi < (\bar{z}_1) } \right)
\]

\[
= (z_2 - z_1)^{\alpha_2 \alpha_1} (\bar{z}_2 - \bar{z}_1)^{\alpha_2 \alpha_1} (z_3 - z_2)^{\alpha_3 \alpha_2} (\bar{z}_3 - \bar{z}_2)^{\alpha_3 \alpha_2} (z_3 - z_1)^{\alpha_3 \alpha_1} (\bar{z}_3 - \bar{z}_1)^{\alpha_3 \alpha_1} \\
\times e^{ \delta(\alpha_1 + \alpha_2 + \alpha_3, 0) }
\]

where we used the formula for OPE of two exponents and once again exchanging relations \([10], [11]\):

\[
e^{-\alpha_3 \varphi < (z_3)} e^{ \alpha_2 \varphi < (z_2) } e^{ \alpha_1 \varphi < (z_1) } = \left( 1 - \frac{z_2}{z_3} \right)^{\alpha_3 \alpha_2} \left( 1 - \frac{z_1}{z_3} \right)^{\alpha_3 \alpha_1} e^{ \alpha_2 \varphi < (z_2) } e^{ \alpha_1 \varphi < (z_1) } e^{-\alpha_3 \varphi < (z_3) }
\]

\[
(z_3 \bar{z}_3)^{\alpha_3 \pi_0} e^{ \delta(\alpha_2 + \alpha_1) \varphi_0 } = e^{ \delta(\alpha_2 + \alpha_1) \varphi_0 } (z_3 \bar{z}_3)^{\alpha_3 \pi_0} (z \bar{z})^{\alpha_3(\alpha_2 + \alpha_1)}
\]

In general n-point function:

\[
\langle E^{\alpha_3}(z_n, \bar{z}_n) \ldots E^{\alpha_2}(z_2, \bar{z}_2) E^{\alpha_1}(z_1, \bar{z}_1) \rangle = \prod_{i>j} (z_i - z_j)^{\alpha_i \alpha_j} (\bar{z}_i - \bar{z}_j)^{\alpha_i \alpha_j} \delta_{\alpha_1 + \alpha_2 + \ldots + \alpha_n, 0}
\]

### 2.4.5 Transformation law for normal ordered exponents

Knowing OPE \( T(z) E^\alpha(w, \bar{w}) \) we can compute variation of the field under the infinitesimal transformation:

\[
\delta_{\epsilon, \bar{z}} E^\alpha(w, \bar{w}) = \int_w \frac{dz}{2\pi i} \epsilon(z) T(z) E^\alpha(w, \bar{w}) + \int_w \frac{dz}{2\pi i} \bar{\epsilon} (\bar{z}) \bar{T}(\bar{z}) E^\alpha(w, \bar{w})
\]

\[
= \int_w \frac{dz}{2\pi i} \epsilon(z) \left[ \frac{\alpha^2 / 2}{(z - w)^2} E^\alpha(w, \bar{w}) + \frac{1}{(z - w)} \partial_w E^\alpha(w, \bar{w}) \right]
\]
\[ + \oint \frac{dz}{2\pi i} \tilde{\epsilon}(\bar{z}) \left[ \frac{\alpha^2/2}{(\bar{z} - \bar{w})^2} E^\alpha(w, \bar{w}) + \frac{1}{(\bar{z} - \bar{w})} \tilde{\partial}_w E^\alpha(w, \bar{w}) \right] \]

\[ = \frac{\alpha^2}{2} E^\alpha(w, \bar{w}) \tilde{\epsilon}(w) + \epsilon(w) \tilde{\partial} E^\alpha(w, \bar{w}) + \frac{\alpha^2}{2} E^\alpha(w, \bar{w}) \tilde{\bar{\epsilon}}(\bar{w}) + \epsilon(\bar{w}) \tilde{\bar{\partial}} E^\alpha(w, \bar{w}) \]

This corresponds to the following global transformation law \((z \to w(z), \bar{z} \to \bar{w}(\bar{z}))\):

\[ E^\alpha(w, \bar{w}) = \left( \frac{\partial w}{\partial z} \right)^{-\Delta_\alpha} \left( \frac{\partial \bar{w}}{\partial \bar{z}} \right)^{-\bar{\Delta}_\alpha} E^\alpha(z, \bar{z}) \]

(12)

where \(\Delta_\alpha\) and \(\bar{\Delta}_\alpha\) are called the holomorphic and antiholomorphic conformal weights. For the field \(E^\alpha(z, \bar{z})\) they are equal:

\[ \Delta_\alpha = \bar{\Delta}_\alpha = \frac{\alpha^2}{2} \]

In general

\[ (L_0 + \bar{L}_0) \phi_{\Delta, \bar{\Delta}} = (\Delta_\alpha + \bar{\Delta}_\alpha) \phi_{\Delta, \bar{\Delta}} = \lambda \phi_{\Delta, \bar{\Delta}} \]

\[ (L_0 - \bar{L}_0) \phi_{\Delta, \bar{\Delta}} = (\Delta_\alpha - \bar{\Delta}_\alpha) \phi_{\Delta, \bar{\Delta}} = s \phi_{\Delta, \bar{\Delta}} \]

where \(\lambda\) is the scaling dimension and \(s\) is the spin of a field. Thus the conformal weights are defined by

\[ \Delta = \frac{1}{2}(\lambda + s), \quad \bar{\Delta} = \frac{1}{2}(\lambda - s). \]

### 2.4.6 Correlation functions from symmetry

- **The 2-point correlation function**

We will transform the fields to 0 and \(\infty\). The first step is to perform transformation \(z_i \to z_i - z_1\). The derivatives in (12) are trivial and we get:

\[ \langle E^{\alpha_2}(z_2, \bar{z}_2) E^{\alpha_1}(z_1, \bar{z}_1) \rangle = \langle E^{\alpha_2}(z_2 - z_1, \bar{z}_2 - \bar{z}_1) E^{\alpha_1}(0, 0) \rangle \]

For the next transformation we choose \(f(z) = \frac{z/(z_2 - z_1)}{(z_2 - z_1) - z}\). The derivative:

\[ f'(z) = \frac{df(z)}{dz} = \frac{1}{((z_2 - z_1) - z)^2}, \quad f'(0) = \frac{1}{(z_2 - z_1)^2}, \quad f'(z_2 - z_1) = \lim_{a \to 0} \frac{1}{a^2} \]

The correlation function transforms as follows (note that we express the r.h.s of (12) and thus we have \([f'(0)]^{\bar{\Delta}_1}[f'(z_2 - z_1)]^{\Delta_2}\):

\[ \langle E^{\alpha_2}(z_2, \bar{z}_2) E^{\alpha_1}(z_1, \bar{z}_1) \rangle = \langle E^{\alpha_2}(z_2 - z_1, \bar{z}_2 - \bar{z}_1) E^{\alpha_1}(0, 0) \rangle \]

\[ = \lim_{a, \bar{a} \to 0} a^{-2\Delta_2} \bar{a}^{-2\bar{\Delta}_2} (z_2 - z_1)^{-2\Delta_1} (\bar{z}_2 - \bar{z}_1)^{-2\bar{\Delta}_1} \left( E^{\alpha_2}(\frac{1}{a}, \frac{1}{\bar{a}}) E^{\alpha_1}(0, 0) \right) \]

\[ = (z_2 - z_1)^{-2\Delta_1} (\bar{z}_2 - \bar{z}_1)^{-2\bar{\Delta}_1} \langle E^{\alpha_2}(\infty, \infty) E^{\alpha_1}(0, 0) \rangle \]
where the field at infinity is defined by the limit:

\[ E^\alpha(\infty, \infty) = \lim_{z \to 0} (z \bar{z})^{-2\Delta} E^\alpha(1/z, 1/\bar{z}) \]

- **The 3-point correlation function**

We will fix 3 locations of the fields using global conformal transformations. In the first step we set one location equal to 0: \( z_i \to z_i - z_1 \)

\[
\langle E^\alpha_3(z_3, \bar{z}_3) E^\alpha_2(z_2, \bar{z}_2) E^\alpha_1(z_1, \bar{z}_1) \rangle = \langle E^\alpha_3(z_{31}, \bar{z}_{31}) E^\alpha_2(z_{21}, \bar{z}_{21}) E^\alpha_1(0, 0) \rangle
\]

where \( z_{ij} = z_i - z_j \).

Next transformation fix the second location: \( z_i \to \frac{z_i}{z_{21}} \)

\[
\langle E^\alpha_3(z_{31}, \bar{z}_{31}) E^\alpha_2(z_{21}, \bar{z}_{21}) E^\alpha_1(0, 0) \rangle = (z_{21}\bar{z}_{21})^{-\Delta_3+\Delta_2+\Delta_1} \left\langle E^\alpha_3 \left( \frac{z_{31}}{z_{21}}, \frac{\bar{z}_{31}}{\bar{z}_{21}} \right) E^\alpha_2(1, 1) E^\alpha_1(0, 0) \right\rangle
\]

Using the last transformation we want to send the third location to infinity, but the locations 0 or 1 have to stay safe. We choose the following function:

\[
f(a, w) = \frac{w}{1 + a(1 - w)}, \quad f'(a, w) \equiv \partial_w f(a, w) = \frac{1 + a}{(1 + a(1 - w))^2}
\]

We define \( a_R \) such that:

\[
f(a_R, w_3) = R, \quad \text{then} \quad f'(a_R, w_3) = \frac{1 + a_R}{(1 + a_R(1 - w_3))^2} = \frac{R^2}{w_3^2} (1 + a_R)
\]

where \( w_3 = \frac{z_{31}}{z_{21}} \). The other two derivatives:

\[
f'(a_R, 0) = \frac{1}{1 + a_R}, \quad f'(a_R, 1) = 1 + a_R
\]

In the \( R \to \infty \) limit:

\[
\lim_{R \to \infty} f(a_R, w_3) = f(a_\infty, w_3) = \infty \quad \Rightarrow \quad a_\infty (1 - w_3) = -1,
\]

thus

\[
a_\infty = \frac{1}{w_3 - 1} = \frac{1}{\frac{z_{31}}{z_{21}} - 1} = \frac{z_{21}}{z_{32}}, \quad a_\infty + 1 = \frac{z_{31}}{z_{32}} + 1 = \frac{z_{31}}{z_{32}}
\]

So the multiplicative factor corresponding to derivatives reads:

\[
\lim_{R, R \to \infty} (f'(a_R, w_3))^{\Delta_3} (f'(a_R, 1))^{\Delta_2} (f'(a_R, 0))^{\Delta_1} = \lim_{R, R \to \infty} \left( \frac{R^2}{w_3^2} \right)^{\Delta_3} (1 + a_\infty)^{\Delta_3+\Delta_2-\Delta_1}
\]

\[
= \lim_{R, R \to \infty} R^{2\Delta_3} \left( \frac{z_{31}}{z_{21}} \right)^{-2\Delta_3} \left( \frac{z_{31}}{z_{32}} \right)^{(\Delta_3+\Delta_2-\Delta_1)}
\]

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And finally the 3-point function:

\[
\langle E^{\alpha_3}(z_{31}, \bar{z}_{31}) E^{\alpha_2}(z_{21}, \bar{z}_{21}) E^{\alpha_1}(0, 0) \rangle \\
= (z_{21}\bar{z}_{21})^{-(\Delta_3 + \Delta_2 + \Delta_1)} \langle E^{\alpha_3}(z_{31}, \bar{z}_{31}) E^{\alpha_2}(1, 1) E^{\alpha_1}(0, 0) \rangle \\
= \lim_{R, \bar{R} \to \infty} (z_{21}\bar{z}_{21})^{-(\Delta_3 + \Delta_2 + \Delta_1)} (z_{31}\bar{z}_{31})^{(\Delta_1 - \Delta_2 - \Delta_3)} \langle E^{\alpha_3}(R, \bar{R}) E^{\alpha_2}(1, 1) E^{\alpha_1}(0, 0) \rangle \\
= (z_{21}\bar{z}_{21})^{(\Delta_3 - \Delta_2 - \Delta_1)} (z_{31}\bar{z}_{31})^{(\Delta_2 - \Delta_3 - \Delta_1)} (z_{32}\bar{z}_{32})^{(\Delta_1 - \Delta_2 - \Delta_3)} \langle E^{\alpha_3}(\infty, \infty) E^{\alpha_2}(1, 1) E^{\alpha_1}(0, 0) \rangle
\]

The 3-point functions of the exponents in standard locations is just a delta function - it is non-zero when the parameters satisfy conservation rule:

\[\langle E^{\alpha_3}(\infty, \infty) E^{\alpha_2}(1, 1) E^{\alpha_1}(0, 0) \rangle = \delta_{\alpha_1 + \alpha_2 + \alpha_3, 0}\]

Using the conservation rule one can rewrite:

\[
(\Delta_3 - \Delta_2 - \Delta_1) = \frac{1}{2}(\alpha_3^2 - \alpha_2^2 - \alpha_1^2) = \frac{1}{2}((\alpha_2 + \alpha_1)^2 - \alpha_2^2 - \alpha_1^2) = \alpha_2 \alpha_1
\]

We can check that the derived formula for the 3-point function is the same as the one calculated in the last subsection.

### 2.5 Twisted boundary conditions

#### 2.5.1 Mode expansion

The free field Lagrangian

\[
\frac{1}{8\pi} \int dx \left[ (\partial_t \Phi)^2 - (\partial_x \Phi)^2 \right]
\]

Consider a different variant of the free scalar theory: we assume **antiperiodic boundary conditions** on the cylinder:

\[\Phi(x + L, t) = -\Phi(x, t)\]

The Fourier mode expansion:

\[
\Phi(x, t) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{2\pi ikx/L} \varphi_k(t)
\]

\[
\varphi_k(t) = \frac{1}{L} \int dx \, e^{-2\pi ikx/L} \Phi(x, t)
\]

The mode expansion of the free field:

\[
\Phi(z, \bar{z}) = i \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{1}{k} (a_k z^{-k} + \bar{a}_k \bar{z}^{-k})
\]
2.5.2 Fock space

There is no zero mode, we have only one vacuum state:

\[ a_k |\sigma_0\rangle = 0, \quad k \in \mathbb{N} - \frac{1}{2} \]

The Fock space is composed from states created from the vacuum state:

\[ a_{-k_1} \ldots a_{-k_m} |\sigma_0\rangle = 0, \quad k \in \mathbb{N} - \frac{1}{2} \]

The inner product is defined by the standard conditions:

\[ a^\dagger_k = a_{-k}, \quad \langle \sigma_0 | \sigma_0 \rangle = 1. \]

The space of states:

\[ \mathcal{H} = \mathcal{F}_{\sigma_0} \otimes \mathcal{F}_{\sigma_0} \]

2.5.3 Conformal weight of the vacuum state

- Vacuum expectation values

From mode expansion of the free field one gets:

\[ \langle \sigma_0 | \phi(z) \partial \phi(w) | \sigma_0 \rangle = \sum_{n,m \in \mathbb{Z} + \frac{1}{2}} \frac{1}{n} \langle \sigma_0 | a_n a_m | \sigma_0 \rangle z^{-n} w^{-m} - 1 \]

\[ = \frac{1}{w} \sum_{n \in \mathbb{N} - \frac{1}{2}} (w/z)^n = \frac{1}{w} \sqrt{w} \frac{1}{w} \frac{1}{z} \frac{1}{w} = \sqrt{w} \frac{1}{w} \frac{1}{z} - w \]

and for the derivative:

\[ \langle \sigma_0 | \partial \phi(z) \partial \phi(w) | \sigma_0 \rangle = -\frac{1}{2} \frac{\sqrt{w} + \sqrt{w}/z}{(z-w)^2} \]

The vacuum expectation value of the energy-momentum tensor:

\[ \langle \sigma_0 | T(z) | \sigma_0 \rangle = -\frac{1}{2} \lim_{\epsilon \to 0} \left[ \langle \sigma_0 | \partial \phi(z + \epsilon) \partial \phi(z) | \sigma_0 \rangle + \frac{1}{\epsilon^2} \right] = \frac{1}{16 z^2} \]

- The zero mode of the energy momentum tensor:

\[ \langle \sigma_0 | L_0 | \sigma_0 \rangle = \frac{1}{2\pi i} \oint dz \; z \langle \sigma_0 | T(z) | \sigma_0 \rangle = \frac{1}{16} \]

We have just calculated the conformal weight of the vacuum state \( \sigma_0 \):

\[ L_0 |\sigma_0\rangle = \frac{1}{16} |\sigma_0\rangle \]

The modes of energy momentum tensor reads:

\[ L_0 = \sum_{k \in \mathbb{N} + \frac{1}{2}} a_{-k} a_k + \frac{1}{16}, \quad L_n = \frac{1}{2} \sum_{k \in \mathbb{Z} + \frac{1}{2}} a_{n-k} a_k \]
3 Free fermion

3.1 Definition

The action of free fermion theory:

\[ S = \frac{1}{4\pi} \int d^2 x \bar{\Psi} \gamma^0 \gamma^\mu \partial_\mu \Psi \]

where the Dirac matrices \( \gamma^\mu \) satisfy algebra: \( \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu} \). Writing the action in terms of spinor components \( \Psi = (\psi, \bar{\psi}) \) one has:

\[ S = \frac{1}{2\pi} \int d^2 x \left( \bar{\psi} \partial \bar{\psi} + \psi \bar{\partial} \psi \right) \]

• The classical equations of motion read

\[ \partial \bar{\psi} = 0, \quad \bar{\partial} \psi = 0 \]

and the solutions are holomorphic \( \psi(z) \) and antiholomorphic \( \bar{\psi}(\bar{z}) \).

• Propagator is a solution of the equation

\[ \bar{\partial} \langle \psi(z)\psi(w) \rangle = 2\pi \delta^{(2)}(z - w) \]

Using the following representation of the delta function:

\[ \delta^{(2)}(z - w) = \frac{1}{2\pi} \bar{\partial} \frac{1}{z - w} \]

one gets

\[ \langle \psi(z)\psi(w) \rangle = \frac{1}{z - w} \]

The same for \( \bar{\psi} \):

\[ \langle \bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) \rangle = \frac{1}{\bar{z} - \bar{w}} \]

After differentiation:

\[ \langle \psi(z)\partial_\mu \psi(w) \rangle = \frac{1}{(z - w)^2} \]

• Energy-momentum tensor

From Noether’s theorem:

\[ T^{\mu\nu} = g^{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \partial_\nu \Phi \]
\[ T^{zz} = 2 \frac{\partial L}{\partial \Phi} \partial \Phi = \frac{1}{\pi} \psi \partial \psi \]
\[ T^{\bar{z}z} = 2 \frac{\partial L}{\partial \Phi} \partial \Phi = \frac{1}{\pi} \bar{\psi} \partial \bar{\psi} \]
\[ T^{z\bar{z}} = 2 \frac{\partial L}{\partial \Phi} \partial \Phi - 2L = -\frac{1}{\pi} \psi \partial \bar{\psi}, \quad T^{\bar{z}\bar{z}} = -\frac{1}{\pi} \bar{\psi} \partial \psi \]

The nondiagonal components vanish if the classical equations of motion are satisfied. The holomorphic and anti-holomorphic components have the following:

\[ T(z) = -2\pi T_{zz} = -\frac{1}{2} \pi T^{zz}, \quad \bar{T}(\bar{z}) = -2\pi T_{\bar{z}z} = -\frac{1}{2} \pi T^{\bar{z}z} \]
\[ T(z) = -\frac{1}{2} : \psi(z) \partial \psi(z) := -\frac{1}{2} \lim_{w \rightarrow z} \left[ \psi(z) \partial \psi(w) - \langle \psi(z) \partial \psi(w) \rangle \right] \quad (13) \]

• OPEs

From Wick’s theorem we can calculate the singular terms in OPE:

\[ T(z) \psi(w) = -\frac{1}{2} : \psi(z) \partial \psi(z) : \psi(w) = \frac{1}{2} \psi(w) \frac{1}{(z-w)^2} + \frac{\partial \psi(w)}{z-w} + \text{reg} \]

where one has to remember about the minus sign appearing when two fermion fields change order (here: contraction of \( \psi(z) \) and \( \psi(w) \)). The conformal weight of the free fermion field is \( \frac{1}{2} \).

The OPE of \( T(z) \):

\[ T(z) T(w) = \frac{1}{4} : \psi(z) \partial \psi(z) : : \psi(w) \partial \psi(w) := \frac{1}{4} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(z)}{z-w} \]

Comparing with the general formula (5) we can see that the central charge of the free fermion theory \( c = \frac{1}{2} \).

• transformation law for \( \psi(z) \)

For infinitesimal transformation \( w = z + \epsilon \):

\[ \delta_{\epsilon} \psi'(w) = \oint_w \frac{dz}{2\pi i} \epsilon(z) T(z) \psi(w) = \oint_w \frac{dz}{2\pi i} \epsilon(z) \left( \frac{1}{2} \psi(w) \frac{1}{(z-w)^2} + \frac{\partial \psi(w)}{z-w} \right) \]
\[ = \frac{1}{2} \psi(w) \partial \epsilon(w) + \epsilon(w) \partial \psi(w) \]

The general form:

\[ \psi'(w) = \left( \frac{dw}{dz} \right)^{-\frac{1}{2}} \psi(z) \quad (14) \]
3.2 Canonical quantization on a cylinder

3.2.1 On a cylinder

The mode expansion

\[ \psi(x) = \sqrt{\frac{2\pi}{L}} \sum_k b_k e^{2\pi ikx/L} \]

where the operators \( b_k \) satisfy anticommutation relations

\[ \{b_p, b_q\} = \delta_{p+q,0} \]

There are two types of boundary conditions:

\[ \psi(x + 2\pi L) = \psi(x), \quad \text{Ramond (R)} \]
\[ \psi(x + 2\pi L) = -\psi(x), \quad \text{Neveu–Schwarz (NS)} \]

The modes \( k \) depend on the case:

- \( k \in \mathbb{Z} \) for periodic case (R)
- \( k \in \mathbb{Z} + \frac{1}{2} \) for antiperiodic case (NS)

The Hamiltonian

\[ H = \frac{2\pi}{L} \left( \sum_{k>0} k \ b_{-k}b_k + \sum_{k>0} k \ b_{-k} \bar{b}_k \right) + E_0 \]

The time-dependent field:

\[ \psi(x, t) = \sqrt{\frac{2\pi}{L}} \sum_k b_k e^{-2\pi k(\tau - ix)/L} \]

3.2.2 On the complex plane

We map the cylinder onto the complex plane:

\[ e^{-2\pi(\tau - ix)/L} = e^{-2\pi \xi/L} \rightarrow z \]

The free fermion field transforms according to the law [14]:

\[ \psi_{pl}(z) = \left( \frac{dz}{d\xi} \right)^{-\frac{1}{2}} \psi_{cyl}(z) = \sqrt{\frac{L}{2\pi z}} \psi_{cyl}(z) \]

Thus the mode expansion of the field on the plane:

\[ \psi(z) = \sum_k b_k \ z^{-k-\frac{1}{2}} \]
Because of the factor $\frac{1}{2}$ coming from the transformation law, the meaning of the two types of boundary conditions has changed – as $z$ is encycling the origin one has

$$
\psi(e^{2\pi i}z) = -\psi(z), \quad \text{Ramond (R)},
$$

$$
\psi(e^{2\pi i}z) = \psi(z), \quad \text{Neveu – Schwarz (NS)}
$$

The NS fields are periodic, while the fields of R type are antiperiodic (double valued on the complex plane).

- **Space of states in NS sector**

Since there in no zero mode, the NS vacuum state is unique:

$$
b_k |0\rangle_{NS} = 0, \quad k \in \mathbb{N} + \frac{1}{2}
$$

The space of states is composed of the tensor product of the left and right Fock spaces:

$$
\mathcal{H}^F_{NS} = \mathcal{F}_{NS} \otimes \overline{\mathcal{F}}_{NS}
$$

- **Space of states in R sector**

In the R sector there is a zero mode $b_0$ which isn’t present in the Hamiltonian. It leads to a degeneracy of vacuum: both states $|+\rangle$, $b_0 |+\rangle$ are annihilated by the operators $b_k$ with $k > 0$. Double action of the zero-mode operator can be read off from anticommutation relation: $b_0^2 = \frac{1}{2}$. We choose the following normalization:

$$
b_0 |+\rangle_R = \frac{1}{\sqrt{2}} | - \rangle_R, \quad b_n |+\rangle_R = 0, \quad n \in \mathbb{N}
$$

The space of states is a tensor product of the left and right Fock spaces:

$$
\mathcal{H}^F_R = \mathcal{F}_R \otimes \overline{\mathcal{F}}_R
$$

- **2-point function in NS sector**

From the mode expansion:

$$
\langle 0 | \psi(z) \psi(w) | 0 \rangle_{NS} = \sum_{p,q \in \mathbb{Z} + \frac{1}{2}} \langle 0 | b_p b_q | 0 \rangle_{NS} z^{-p-q} w^{p} = \sum_{p \in \mathbb{N} - \frac{1}{2}} z^{-p-q} w^p
$$

$$
= \frac{1}{z} \sum_{n=0}^{\infty} \left( \frac{w}{z} \right)^n = \frac{1}{z} \frac{1}{1 - w/z} = \frac{1}{z - w}
$$
2.2-point function in R sector

\[ \langle + | \psi(z) \psi(w) | + \rangle_R = \sum_{p,q \in \mathbb{Z}} \langle + | b_p b_q | + \rangle_R z^{-p-\frac{1}{2}} w^{-q-\frac{1}{2}} = \frac{1}{2} \sqrt{z/w} + \sum_{p=1}^{\infty} z^{-p-\frac{1}{2}} w^{p-\frac{1}{2}} \]

\[ = \frac{1}{\sqrt{z/w}} \left( \frac{1}{2} + \sum_{p=1}^{\infty} \left( \frac{w}{z} \right)^p \right) = \frac{1}{2} \sqrt{z/w + \sqrt{w/z}} \]

3.2.3 Virasoro generators

Modes of the energy-momentum tensor:

\[ T(z) = -\frac{1}{2} : \psi(z) \partial \psi(z) : \]

\[ = \frac{1}{2} \sum_{p,q} (q + \frac{1}{2}) z^{-p-\frac{1}{2}} z^{-q-\frac{1}{2}} : b_p b_q : = \frac{1}{2} \sum_{n,q} (q + \frac{1}{2}) z^{-n-2} : b_{n-q} b_q : \]

We can compute the vacuum expectation values of the energy-momentum tensor in both sectors:

\[ \langle 0 | T(z) | 0 \rangle_{NS} = -\frac{1}{2} \lim_{\epsilon \to 0} \left[ \langle 0 | \psi(z + \epsilon) \partial \psi(z) | 0 \rangle_{NS} - \frac{1}{\epsilon^2} \right] = 0 \]

\[ \langle + | T(z) | + \rangle_R = -\frac{1}{2} \lim_{\epsilon \to 0} \left[ \langle + | \psi(z + \epsilon) \partial \psi(z) | + \rangle_R - \frac{1}{\epsilon^2} \right] = \frac{1}{16 z^2} \]

This equations imply that the conformal weights are equal to zero and \( \frac{1}{16} \) for the NS and R vacuum respectively. Thus the zero mode of the energy momentum tensor reads:

\[ L_0 = \sum_{k>0} k b_{-k} b_k \quad (NS : k \in \mathbb{Z} + \frac{1}{2}) \]

\[ L_0 = \sum_{k>0} k b_{-k} b_k + \frac{1}{16} \quad (R : k \in \mathbb{Z}) \]

The Virasoro generators can be written in the following form:

\[ L_n = \frac{1}{2} \sum_{q} (q + \frac{1}{2}) : b_{n-q} b_q : , \]

where the summation index \( q \) runs through half-integers in NS case and integers in R case.