4d Spin foam models of quantum gravity

John Barrett

School of Mathematical Sciences
University of Nottingham

June 30, 2009

Spin foam models

- Quantum gravity without matter
- Models
- Area $= G\hbar$
- Discrete structure at Planck scale (superpositions)
- Continuum picture in $G\hbar \to 0$ limit
- Discreteness compatible with symmetries (c.f. angular momentum)

History

- Ponzano, Regge 1968 (3d gravity spin foam)
- Witten 1989 (3d gravity functional integral)
- Ooguri 1990 (4d spin foam)
- JWB, Crane 1996 (4d gravity spin foam)
- Barbieri 1997 (quantum tetrahedron)
- Engle, Pereira, Rovelli, Livine, Freidel, Krasnov 2007 (4d gravity with Immirzi)
- JWB, Dowdall, Fairbairn, Gomes, Hellmann 2008/9 (asymptotics with Immirzi)

Quantum tetrahedron

Label set (spins)

$$\mathcal{L} = \mathsf{Irrep}(\mathrm{SU}(2)) \cong \{0, \frac{1}{2}, 1, \ldots\}$$

State space

$$\begin{aligned}\mathcal{H}_\Delta &= \mathsf{Inv}(j_1 \otimes j_2 \otimes j_3 \otimes j_4)\\ &= \text{Geometric quantisation of } 2d \text{ phase space } S\end{aligned}$$

S = Euclidean tetrahedron with face areas j_1, j_2, j_3, j_4.

Tetrahedron geometry

Classical model

$$j \in \mathcal{L} \quad \longrightarrow \quad v \in \mathbb{R}^3, |v| = j$$

$$\mathrm{Inv}(j_1 \otimes j_2 \otimes j_3 \otimes j_4) \quad \longrightarrow \quad v_1 + v_2 + v_3 + v_4 = 0$$

Faces of oriented tetrahedron

$$\begin{aligned} \dim \mathcal{H}_\Delta \quad &= \quad \mathrm{vol}(S) \\ \text{Coherent states} \quad &\leftrightarrow \quad \text{Points in } S \end{aligned}$$

State space for hypersurface

Triangulated closed 3-manifold Σ

Labelling

$$j : \text{triangles} \longrightarrow \mathcal{L}$$

$$\mathcal{H}_\Sigma = \bigoplus_j \bigotimes_\Delta \mathcal{H}_\Delta$$

4-simplex amplitude

$\sigma =$ 4-simplex $\qquad \partial\sigma =$ 5 tetrahedra

$$\mathcal{H}_{\partial\sigma} = \bigoplus_{j} \mathcal{H}^1_\Delta \otimes \mathcal{H}^2_\Delta \otimes \mathcal{H}^3_\Delta \otimes \mathcal{H}^4_\Delta \otimes \mathcal{H}^5_\Delta$$

Partition function

$$Z_\sigma : \mathcal{H}_{\partial\sigma} \to \mathbb{C}$$

$$\alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \epsilon \to$$

spin network: 15j-symbols

Ooguri model

Triangulated 4-manifold M
Labelling l:

$$\text{triangles} \to \mathcal{L}$$
$$\text{tetrahedra} \to \text{basis of } \mathcal{H}_\Delta$$

Partition function

$$Z_M = \sum_l \prod_\sigma Z_\sigma \prod_\Delta \dim j$$

- Function of fixed boundary data $\mathcal{H}_{\partial M} \to \mathbb{C}$
- Regularisation

Asymptotics of Ooguri

- Pick $\alpha, \beta, \gamma, \delta, \epsilon$ to be coherent states with definite tetrahedral geometries
- Data is 'Regge-like' if these glue together to give 3-metric
- Spins $\lambda j_1, \lambda j_2, \ldots \lambda j_{10}$, with $\lambda \to \infty$

Then

$$
\begin{aligned}
Z_\sigma &\sim c e^{i\lambda S_R} + c' e^{-i\lambda S_R} \quad \text{if } \partial\sigma = \partial\text{Euclidean 4-simplex} \\
&\sim d e^{i\lambda S'} \quad \text{degenerate geometry} \\
&\sim 0 \quad \text{else}
\end{aligned}
$$

- $S_R = \text{Regge action} = \sum j\Theta$
- Glues to flat manifold

General G

G = group or Hopf algebra

$$\mathcal{L} \subset \mathrm{Irrep}(G)$$

$$\mathcal{H}_\Delta \subset \mathrm{Inv}_G(l_1 \otimes l_2 \otimes l_3 \otimes l_4)$$

Crane-Yetter: $G = \mathrm{U}_q\mathrm{sl2}, q = e^{i\pi/r}$

Relativistic spin network, $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$

$$\mathrm{Irrep}(G) = \{(j, j')\}$$

$$\mathcal{L} = \{(j, j)\}$$

$$\mathcal{H}_\Delta = \text{canonical vertex}$$

- Irreps are bivectors in $\mathbb{R}^4$
- Labels are simple bivectors, area j
- $\dim \mathcal{H}_\Delta = 0$ or 1
- Tetrahedron has area geometry, no unique metric
- Asymptotics $Z_\sigma \sim \sum_{\text{metrics}} c e^{iS_R} + \bar{c} e^{-iS_R} + d$
- gluing of area geometries
- d dominates

Relativistic spin network, $G = \mathrm{SO}(3,1)$

$$\mathrm{Irrep}(G) = \{(k,p) \quad k \in \frac{1}{2}\mathbb{Z}^{\geq 0}, p \in \mathbb{R}\}$$

$$\mathcal{L} = \{(0,p) \quad p \geq 0\}$$

$$\mathcal{H}_\Delta = \text{canonical vertex}$$

- Irreps are spacelike bivectors in Minkowski space
- Labels are simple bivectors, area p
- $\dim \mathcal{H}_\Delta = 0$ or 1
- Tetrahedron has area geometry, no unique metric
- Asymptotics $Z_\sigma \sim \sum_{\text{metrics}} c e^{iS_L} + \bar{c} e^{-iS_L} + d$
- S_L = Lorentzian Regge action
- Again d dominates, gluing of area geometries

EPRL, model with Immirzi , $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$

$$\mathcal{L} = \{\left(\frac{1}{2}(1+\gamma)k, \frac{1}{2}|1-\gamma|k\right) \quad k \in \frac{1}{2}Z^{\geq 0}\}$$

Three-dimensional subgroup

$$\mathrm{SU}(2) \to G$$

$$k \to (\frac{1}{2}(1+\gamma)k, \frac{1}{2}|1-\gamma|k)$$

$$\mathcal{H}_\Delta = \mathrm{Inv}_G(\mathrm{Inv}_{\mathrm{SU}(2)}(k_1 \otimes k_2 \otimes k_3 \otimes k_4))$$

- Area of triangle k, discrete
- Boundary data is quantum tetrahedron
- Gluing now of 3d Euclidean metric geometries

Asymptotics of EPRL

$$Z_\sigma \sim c e^{i\lambda\gamma S_R} + c' e^{-i\lambda\gamma S_R} + a e^{i\lambda S_R} + a' e^{-i\lambda S_R}$$

if $\partial\sigma = \partial$Euclidean 4-simplex

$$\sim d e^{i\lambda S'}$$

if degenerate geometry

$$\sim 0$$

else

Lorentzian model with Immirzi , $G = \mathrm{SO}(3,1)$

$$\mathcal{L} = \{(k, p = \gamma k) \quad k \in \frac{1}{2} Z^{\geq 0}\}$$

Three-dimensional subgroup

$$\mathrm{SU}(2) \to G$$

$$k \to (k, \gamma k)$$

$$\mathcal{H}_\Delta = \mathrm{Inv}_G(\mathrm{Inv}_{\mathrm{SU}(2)}(k_1 \otimes k_2 \otimes k_3 \otimes k_4))$$

- Regularisation
- Area of triangle k, discrete
- Boundary data is quantum tetrahedron
- Gluing again of 3d Euclidean metric geometries

Asymptotics of Lorentzian model with Immirzi

$$Z_\sigma \sim c e^{i\lambda\gamma S_L} + c' e^{-i\lambda\gamma S_L}$$

if $\partial\sigma = \partial$Lorentzian 4-simplex

$$\sim a e^{i\lambda S_R} + a' e^{-i\lambda S_R}$$

if $\partial\sigma = \partial$Euclidean 4-simplex

$$\sim d e^{i\lambda S'}$$

if degenerate geometry

$$\sim 0$$

else