Gravity as constrained BF theory

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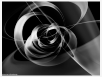
Content of the talk

1. **MacDowell-Mansouri theory**
2. **BF theory reformulation**
3. **Supergravity**
4. **Canonical analysis**



General Relativity

VARIABLE: $g_{\mu\nu}$ metric

Hilbert-Einstein Action

ACTION: $S = \dfrac{1}{2G} \int \sqrt{-g}\,(R - 2\Lambda)\,\mathrm{d}^4 x$

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 0, \qquad R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

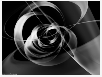
VARIABLE: *so*$(3,1)$-*connection* ω^{ij}_{μ} and *tetrad* e^{i}_{μ}

Palatini Action

ACTION: $S = \dfrac{1}{G} \int d^4 x\, \epsilon^{ijkl} (R_{\mu\nu\,ij}\, e_{\rho\,k}\, e_{\sigma\,l} - \dfrac{\Lambda}{3} e_{\mu\,i}\, e_{\nu\,j}\, e_{\rho\,k}\, e_{\sigma\,l}) \epsilon^{\mu\nu\rho\sigma}$

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij} + \omega_\mu{}^{i}{}_{k}\, \omega_\nu{}^{kj} - \omega_\nu{}^{i}{}_{k}\, \omega_\mu{}^{kj}, \qquad R^{i}{}_{j} = d\omega^{i}{}_{j} + \omega^{i}{}_{k} \wedge \omega^{k}{}_{j}$$

$$T_{\mu\nu}{}^{i} = \mathcal{D}^{\omega}_{\mu}\, e_\nu{}^{i} - \mathcal{D}^{\omega}_{\nu}\, e_\mu{}^{i}, \qquad T^{i} = D^{\omega} e^{i} = de^{i} + \omega^{i}{}_{j} \wedge e^{j}$$



A_μ^{IJ} connection of the $SO(4,1)$

MacDowell and Mansouri proposal

cosmological constant

$$A^{IJ} = \begin{pmatrix} \omega^{ij} & \frac{1}{\ell} e^i \\ -\frac{1}{\ell} e^j & 0 \end{pmatrix} \qquad \text{where} \qquad \frac{1}{\ell^2} = \frac{\Lambda}{3}$$

where $IJ = 1, 2, ..5$, and $i, j = 1, 2, ..4$; and ℓ is a fundamental length.

With A^{IJ} we obtain $SO(4,1)$ curvature 2-form,

$$F^{IJ} = dA^{IJ} + A^{IK} \wedge A_K^{\ J} = \begin{pmatrix} R^{ij} - \frac{1}{\ell^2} e^i \wedge e^j & \frac{1}{\ell} T^i \\ -\frac{1}{\ell} T^j & 0 \end{pmatrix}$$

$$A_\mu^{IJ} \rightarrow \begin{cases} A_\mu^{ij} = \omega_\mu^{ij} \\ A_\mu^{i5} = \frac{1}{\ell} e_\mu^i \end{cases}, \quad F_{\mu\nu}^{IJ} \rightarrow \begin{cases} F_{\mu\nu}^{ij} = R_{\mu\nu}^{ij} - \frac{1}{\ell^2}(e_\mu^i e_\nu^j - e_\nu^i e_\mu^j) \\ F_{\mu\nu}^{i5} = \frac{1}{\ell} T_{\mu\nu}^i \end{cases}$$

MacDowell-Mansouri 1977

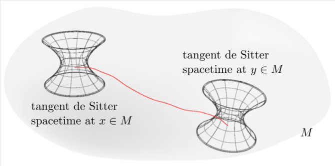
General Relativity as gauge symmetry breaking theory

$$S_{MM}[A] = -\frac{3}{2G\Lambda}\int tr\left(\hat{F}\wedge\star\hat{F}\right) = S_{Palatini} + \frac{3}{2G\Lambda}\int tr\left(R\wedge\star R\right)$$

$$S_{MM}[A] = -\frac{3}{2G\Lambda}\int \left(F_{ij}\wedge F_{kl}\epsilon^{ijkl}\right) = S_{Palatini} + \frac{3}{2G\Lambda}\int R_{ij}\wedge R_{kl}\epsilon^{ijkl}$$

Geometric interpretation

$SO(4,1)$ connection $A = (\omega, e)$ encodes the geometry of spacetime $\mathcal{M}$ by "rolling de Sitter spacetime along $\mathcal{M}$" [D.Wise]



Perturbed BF theory

Introduce independent $so(4,1)$-valued $2-form\ B$ to the action

$$\mathcal{S} = \int_{\mathcal{M}} tr\left(B \wedge F - \frac{G\Lambda}{6}\hat{B} \wedge *\hat{B}\right)$$

In this form Macdowell-Mansouri gravity has the appearance of a "deformation" of a topological gauge theory.

The symmetry breaking occurs in the last term, with a dimensionless coefficient proportional to: $G\Lambda \sim 10^{-120}$.

In some sense general relativity is "not too far pertubatively" from a topological field theory.

Holst action 1996

Ashtekar connection

Phase space of gravity can be described by self and antiselfdual connections provided by projector $\frac{1}{2}(1 \mp i\star)$

$$ {}^{\pm}\omega_a^i = \omega_a^{0i} \mp \frac{i}{2}\epsilon^{ijk}\omega_{a\,jk} $$

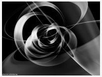
Real Barbero-Immirzi connection

$$ {}^{-}\omega_a^i = \omega_a^{0i} + \gamma\epsilon^{0ijk}\omega_{a\,jk}, \qquad \mathcal{P}_a^i = \frac{4}{G}\epsilon^{ijk}\epsilon_{abc}\,e_j^b\; e_k^c $$

$$ \{{}^{-}\omega_a^i(x), \mathcal{P}_b^j(y)\} = \gamma\delta(x-y)\delta^{ij}\delta_{ab} $$

Holst Action

$$ S = \frac{1}{G}\int \epsilon^{ijkl} R_{\mu\nu\,ij}\, e_{\rho\,k}\, e_{\sigma\,l} + \frac{2}{G\gamma}\int R_{\mu\nu\,ij}\, e_\nu^i\, e_\rho^j\; \epsilon^{\mu\nu\rho\sigma} $$



Starodubtsev and Freidel

Connection A is supplemented by an independent locally $so(4,1)$-valued 2-form B:

$$\mathcal{S} = \int_{\mathcal{M}} tr\left(B \wedge F - \frac{\beta}{2} B \wedge B - \frac{\alpha}{4} \hat{B} \wedge * \hat{B}\right)$$

Action proposed by Starodubtsev and Freidel

$$\mathcal{S} = \int_{\mathcal{M}} \left(B_{IJ} \wedge F^{IJ} - \frac{\beta}{2} B_{IJ} \wedge B^{IJ} - \frac{\alpha}{4} \epsilon_{ijkl5} B^{ij} \wedge B^{kl}\right)$$

with constants

$$\gamma = \frac{\beta}{\alpha}, \quad \Lambda = \frac{3}{\ell^2}, \quad \alpha = \frac{G\Lambda}{3(1+\gamma^2)}$$

Constrained BF theory

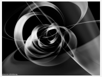
$$\mathcal{S} = \int d^4x\, \epsilon^{\mu\nu\lambda\rho} \left(B_{\mu\nu IJ} F^{IJ}_{\lambda\rho} - \frac{\beta}{2} B_{\mu\nu IJ} B^{IJ}_{\lambda\rho} - \frac{\alpha}{4} \epsilon_{IJKL5} B^{IJ}_{\mu\nu} B^{KL}_{\lambda\rho} \right)$$

$$\begin{aligned} S &= \frac{1}{G} \int \epsilon^{ijkl} (R_{\mu\nu\, ij}\, e_{\rho\, k}\, e_{\sigma\, l} - \frac{\Lambda}{3} e_{\mu\, i}\, e_{\nu\, j}\, e_{\rho\, k}\, e_{\sigma\, l}) \epsilon^{\mu\nu\rho\sigma} \\ &+ \frac{2}{G\gamma} \int R_{\mu\nu\; ij}\, e^{\;i}_{\nu}\, e^{\;j}_{\rho}\; \epsilon^{\mu\nu\rho\sigma} + \frac{\gamma^2+1}{\gamma\, G} NY_4 + \frac{3\gamma}{2G\Lambda} P_4 - \frac{3}{4G\Lambda} E_4 \end{aligned}$$

The last three terms are proportional to topological invariants (Nieh-Yan, Ponryagin, and Euler):

$$NY_4 = \int (T_{\mu\nu\; i}\, T^{i}_{\rho\sigma} - 2\, R_{\mu\nu\; ij}\, e^{\;i}_{\nu}\, e^{\;j}_{\rho})\, \epsilon^{\mu\nu\rho\sigma}\ ,$$

$$P_4 = \int R_{\mu\nu\; ij}\, R^{ij}_{\rho\sigma}\, \epsilon^{\mu\nu\rho\sigma}, \qquad E_4 = \int R_{\mu\nu\; ij}\, R_{\rho\sigma\; kl} \epsilon^{ijkl}\, \epsilon^{\mu\nu\rho\sigma}$$



Supergravity $\mathcal{N} = 1$

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Supergravity as a constrained *BF* theory

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Adding gravitino ψ^α_μ spin-3/2 field

$$\mathbb{A}_\mu = \frac{1}{2}\omega_\mu{}^{ij} M_{ij} + \frac{1}{\ell} e_\mu{}^{i} P_i + \kappa \bar{\psi}^\alpha_\mu Q_\alpha$$

Curvature

$$\mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu - i[\mathbb{A}_\mu, \mathbb{A}_\nu] = \frac{1}{2} F^{(s)IJ}_{\mu\nu} M_{IJ} + \bar{\mathcal{F}}^\alpha Q_\alpha$$

Supergravity $\mathcal{N} = 1$

Supergravity action

$$S^{sugra} = \frac{1}{G} \int d^x \epsilon^{\mu\nu\rho\sigma} \epsilon_{ijkl} \left(R_{\mu\nu}{}^{ij}\, e_\rho{}^k\, e_\sigma{}^l - \frac{\Lambda}{3}\, e_\mu{}^i\, e_\nu{}^j\, e_\rho{}^k\, e_\sigma{}^l \right)$$

$$-\epsilon^{\mu\nu\rho\sigma} \left(\frac{1}{2}\, \bar{\psi}_\mu\, \gamma_5\, \gamma_i\, e^i_\nu D^\omega_\rho \psi_\sigma \, - \frac{i}{4\ell}\, \bar{\psi}_\mu\, \gamma_5\, \gamma_{ij}\, e^i_\nu e^j_\rho\, \psi_\sigma \right)$$

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Cosmological constant in supergravity

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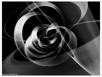
We construct an extension of pure supergravity which contains a cosmological term and a masslike term for the spin-3/2 field. Unlike another recent model which incorporates these features, that presented here is constructed from the usual spin-2, spin-3/2 fields alone.

The action

$$I = \int \left(-\frac{1}{4\kappa^2}\, e e^{a\mu} e^{b\nu} R_{\mu\nu ab} + \frac{3\lambda^2 e}{2} - \tfrac{1}{2}\, \epsilon^{\lambda\rho\mu\nu} \bar{\psi}_\lambda \gamma_5 \gamma_\mu D_\nu \psi_\rho - \lambda\kappa e \bar{\psi}_\lambda \sigma^{\lambda\rho} \psi_\rho \right) d^4x$$



Canonical analysis

$$S = \int dt \int d^3x \left(\dot{A}_i^{IJ} \mathcal{P}^i_{IJ} + \boxed{A_0^{IJ}} (\mathcal{D}_i^A \mathcal{P}^i_{IJ}) + \right.$$
$$\left. + \boxed{{B_{0i}}^{IJ}} \left[2\epsilon^{ijk} F_{jk\,IJ} - \beta\,\mathcal{P}^i_{IJ} - \frac{\alpha}{2}\,\epsilon_{IJKL4}\,\mathcal{P}^{i\,KL} \right] \right)$$

$$\{A_i^{IJ}(x), P^j_{KL}\} = \delta^{IJ}_{KL}\delta(x-y)\delta^j_i \qquad \text{where } \mathcal{P}^{i\,IJ} \equiv 2\epsilon^{ijk}\,{B_{jk}}^{IJ}$$

Hamiltonian analysis of SO(4, 1) constrained BF theory

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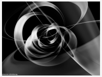
(Dated: March 12, 2010)

In this paper we discuss canonical analysis of SO(4, 1) constrained BF theory. The action of this theory contains topological terms appended by a term that breaks the gauge symmetry down to the Lorentz subgroup of SO(3, 1). The equations of motion of this theory turn out to be the vacuum Einstein equations. By solving the B field equations one finds that the action of this theory contains not only the standard Einstein-Cartan term, but also the Holst term proportional to the inverse of the Immirzi parameter, as well as a combination of topological invariants. We show that the structure of the constraints of a SO(4, 1) constrained BF theory is exactly that of gravity in Holst formulation. We also briefly discuss quantization of the theory.



Canonical analysis

Complicated explicit constraints

$$
\begin{aligned}
\Phi^i_\alpha &= \mathcal{P}^i_\alpha - \frac{4}{\ell\beta}\epsilon^{ijk}\,\mathcal{D}^\omega_j\, e_{k\ \alpha} \approx 0 \\
\Phi^i_{\alpha\beta} &= \mathcal{P}^i_{\alpha\beta} - M_{\alpha\beta}{}^{\gamma\delta}\, F_{jk\ \gamma\delta}\,\epsilon^{ijk} \approx 0 \\
\Pi_{\alpha\beta} &= \frac{2}{\ell^2}\,\epsilon^{ijk}\mathcal{D}^\omega_i\left(K_{\alpha\beta}{}^{\gamma\delta}\ e_{j\ \gamma}\, e_{k\ \delta}\right) \approx 0 \\
\Pi_\alpha &= \frac{1}{\ell}\,\epsilon^{ijk}\ K_{\alpha\beta}{}^{\gamma\delta}\ e_i{}^\beta\ R_{jk\ \gamma\delta} - \frac{2\alpha}{(\alpha^2+\beta^2)\ell^3}\,\epsilon^{ijk}\,\epsilon_{\alpha\beta\gamma\delta}\ e_i{}^\beta\ e_j{}^\gamma\ e_k{}^\delta \approx 0
\end{aligned}
$$

where

$$M^{\alpha\beta}{}_{\gamma\delta} \equiv \frac{\alpha}{(\alpha^2+\beta^2)}(\gamma\,\delta^{\alpha\beta}_{\gamma\delta} - \epsilon^{\alpha\beta}{}_{\gamma\delta}),$$

$$K^{\alpha\beta}{}_{\gamma\delta} \equiv \frac{\alpha}{(\alpha^2+\beta^2)}(\tfrac{1}{\gamma}\,\delta^{\alpha\beta}_{\gamma\delta} + \epsilon^{\alpha\beta}{}_{\gamma\delta})$$

Canonical analysis

Topological part of the action

$$S_{topological} = \frac{\gamma^2+1}{\gamma\, G} N Y_4 + \frac{3\gamma}{2G\Lambda} P_4 - \frac{3}{4G\Lambda} E_4$$

$$S_{topological} = \int \partial_\mu(\ something\ )$$

For constant time surfaces, a manifold $\Sigma \times \mathrm{R}$ without boundary ($\partial\Sigma = 0$), all terms with spacial derivatives drop out, and

$$S_{topological} = \int \partial_0 W(e,\omega)$$

where $W(\omega, e)$ is a functional of torsion and self and anti-self dual Chern-Simons forms $\mathcal{L}_{CS}$:

$$W = \frac{4}{\beta\ell^2}\int_\Sigma \epsilon^{ijk}\left(e_{i\ \alpha}\mathcal{D}^\omega_j\, e_k{}^\alpha\right) + \frac{2\alpha}{(\alpha^2+\beta^2)}\int_\Sigma \Big((\gamma - i)\mathcal{L}_{CS}({}^+\omega) + (\gamma + i)\mathcal{L}_{CS}({}^-\omega)\Big)$$

Canonical transformation

$$\mathbf{P}\cdot\dot{\mathbf{Q}} - K(\mathbf{Q},\mathbf{P},t) + \frac{dW}{dt} = [\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)]$$

New momenta

$$\boxed{\mathscr{P}^i_\alpha} = \mathcal{P}^i_\alpha + \{\mathcal{P}^i_\alpha, W(\omega,e)\}, \quad \boxed{\mathscr{P}^i_{\alpha\beta}} = \mathcal{P}^i_{\alpha\beta} + \{\mathcal{P}^i_{\alpha\beta}, W(\omega,e)\}$$

Canonical analysis of constrained BF theory for manifold without boundary

is equivalent to analysis of

Holst action with shifted definition of the momenta

Noether charge

Action

$$S = \frac{1}{32\pi G}\int \epsilon_{abcd}(R^{ab} - \frac{\Lambda}{3}e^a \wedge e^b)\wedge e^c \wedge e^d + \kappa \int \mathcal{E}_4$$

For Schwarzschild-AdS black hole

$$Q(\partial_t) = \frac{M}{2}\left(1 + \frac{64\pi G}{\ell^2}\kappa\right) + \lim_{r\to\infty}\frac{\pi r^3}{4G\ell^2}\left(1 - \frac{64\pi G}{\ell^2}\kappa\right)$$

For κ exactly from MacDowell-Mansouri model:

$$S_{MM}[A] = \frac{-3}{64\pi G\Lambda}\int (R_{ij} - \frac{1}{\ell^2}e_i \wedge e_j)\wedge(R_{kl} - \frac{1}{\ell^2}e_k \wedge e_l)\epsilon^{ijkl}$$

cures the result.

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