GRAVITATIONAL CONSTANT AND TORSION

by Prasanta Mahato

Abstract:

Riemann-Cartan space-time is considered here. It has been shown that when we link topological Nieh-Yan density with the gravitational constant, we then obtain Einstein-Hilbert Lagrangian as a consequence.

Reviewed by Remigiusz Durka

Table of content

- Torsion and Curvature
- Review of some topological aspects of Riemann-Cartan theory
- MacDowell-Mansouri theory

- A generic Riemann manifold is equipped with two fundamental and independent entities:

metric
$g_{\mu\nu}$

affine connection
$\Gamma^{\lambda}_{\mu\nu}$

- If the affine connection is not assumed to be a function of the metric, then the local geometry is endowed with two independent tensors: curvature and torsion.

- Cartan's structural equations for a Riemann-Cartan space-time:

curvature
$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$

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where respectively:

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Curvature and Torsion

Torsion appears rather naturally in the commutator of two covariant derivatives for the group of diffeomorphisms of a manifold in a coordinate basis

$$[\nabla_\mu, \nabla_\nu]V^A = -T^\lambda{}_{\mu\nu}\nabla_\lambda V^A + R^A{}_{B\mu\nu}V^B$$

where V^A represents any tensor (or spinor) under diffeomorphisms (or under the group of tangent rotations), and $R^A{}_B$ is the curvature tensor in the corresponding representation.

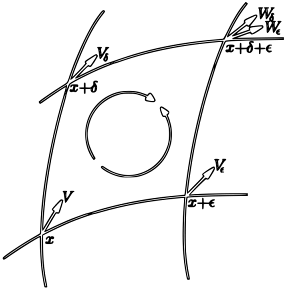
Here curvature and torsion play quite different roles:

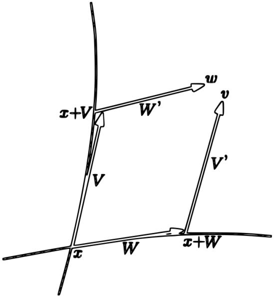
- $T^\lambda{}_{\mu\nu}$ is the structure function for the diffeomorphism group
- $R^A{}_{B\mu\nu}$ is central charge.

This indicates that the gauge approach of gravity can be achieved through the local version of the Lorentz-Poincaré symmetry where $T^\lambda{}_{\mu\nu}$ and $R^A{}_{B\mu\nu}$ represent in the tangent space

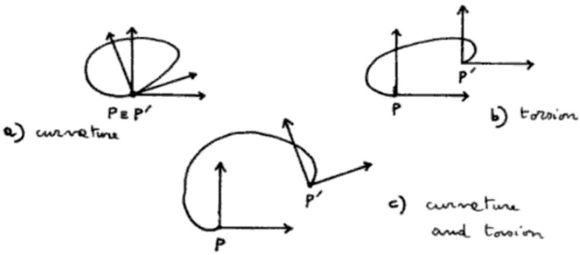
- translational symmetry
- rotational symmetry

Curvature and Torsion





In order to esplain the geometric meaning of the antisymmetric part of the connection, it must be noted that the torsion is related to the translation of a vector, like curvature is related to the rotation, when a vector is displaced along an infinitesimal path in a Riemann-Cartan manifold



<u>Fig.10.1</u>. When a vector describes an infinitesimal closed path, and when this path is developed in the flat space tangent to the manifold, we have a rotation (if there is only curvature) or a translation (if there is only torsion) or both (if there is cufvature and torsion).

Curvature plays an important role in the characterization of the topological structure of the manifold. We can use it to build special objects.

The integral of the second Chern class (also known as Pontryagin number) and Euler number, defined by

$$\mathscr{P}_4 = \frac{1}{8\pi^2}\int_{M_4} R^{ab}{}_{\wedge}R_{ab}, \qquad \textit{where} \qquad \mathscr{E}_4 = \frac{1}{2(4\pi)^2}\int_{M_4} \epsilon_{abdc}R^{ab}{}_{\wedge}R^{cd}$$

are well known examples of topological invariants in four dimensions.

It is a remarkable result of differential geometry that certain global features of a manifold are determined by some local functionals of its intrinsic geometry because main feature of these expressions is that, although they are defined purely in terms of local functions, they only take integer values.

Now, $\mathscr{P}_4$ and $\mathscr{E}_4$ are continuous functionals of the curvature and therefore they do not change under continuous deformations of the geometry of M. Thus, these characteristic classes label topologically distinct four-geometries.

Values of these invariants depend on the global properties of the manifold and are expected to be related to the global values of some physical observables.

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There is less known invariant called Nieh-Yan density N given by

$$N = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b$$

Note that Second Bianchi Identity says:

$$R_{ab} \wedge e^b = DT_a$$

When the vierbein is well defined, we easily see that Nieh-Yan density is the total derivative of a "Chern-Simon like" term.

$$N = T^a \wedge T_a - e^a \wedge DT_a = d(e^a \wedge T_a)$$

So E, P and N are four forms and

- $\mathscr{E} = \int E\, d^4x$
- $\mathscr{P} = \int P\, d^4x$
- $\mathscr{N} = \int N\, d^4x$

are diffeomorphism invariant quantities representing certain global features of the manifold.

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Chandia i Zanelli have shown that if we combine the spin connectionan and the vierbein together in a connection for $SO(5)$ in the form:

$$A^{AB} = \begin{bmatrix} \omega^{ab} & \frac{1}{l}e^a \\ -\frac{1}{l}e^b & 0 \end{bmatrix}$$

where $a, b = 1, 2, ..4; A, B = 1, 2, ..5$ and l is a fundamental length constant, then we obtain the $SO(5)$ curvature 2-form,

$$F^{AB} = dA^{AB} + A^{AC} \wedge A_C^{\ B}$$

$$F^{AB} = \begin{bmatrix} R^{ab} - \frac{1}{l^2}e^a \wedge e^b & \frac{1}{l}T^a \\ -\frac{1}{l}T^b & 0 \end{bmatrix}$$

and the $SO(5)$ Pontryagin density,

$$F^{AB} \wedge F_{AB} = R^{ab} \wedge R_{ab} + \frac{2}{l^2}[T^a \wedge T_a - R^{ab} \wedge e_a \wedge e_b]$$

The first term of the right hand side is the $SO(4)$ Pontryagin density and hence we can write,

$$\frac{2}{l^2}\int_{M_4} N = P_4[SO(5)] - P_4[SO(4)]$$

$$= \frac{1}{(2\pi)^2\ell^2} \times (z_1 + z_2 + z_3), \quad z_i \varepsilon Z$$

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N is an invariant under Lorentz rotation in tangent space but not in the case of (A)dS boost there i.e. when $\frac{1}{\ell}e^a$ itself transforms as a gauge field.

There is a known ***Lemma*** which states that:

For $d = 4k$, the only d-forms built from e^a, R^{ab} and T^a, invariant under AdS group, are the Chern characters for $SO(d+1)$.

This lemma is equally valid in case of Lorentz group $SO(d-1,1)$ when the (A)dS group is either $SO(d,1)$, $SO(d-1,2)$ or $SO(d+1)$, because the analysis that follows is insensitive of the signature. For $d = 4$ there is only one such AdS invariant, the second Chern character of the AdS group, given by $F^{AB} \wedge F_{AB}$.

Hence none of P and N is AdS invariant only their combination is (A)dS invariant.

$$P_4(SO(5-i,i))_{i=1 \text{ or } 2} = P_4 + \frac{2}{\ell^2}N_4$$

Formulation of General Relativity as gauge symmetry breaking theory

$SO(3,1)$	$SO(4,1)$

e^i tetrad

ω^{ij} $so(3,1)$-connection

$$A^{IJ}_\mu \to \begin{cases} A^{ij}_\mu = \omega^{ij}_\mu \\ A^{i5}_\mu = \frac{1}{\ell} e^i_{\ \mu} \end{cases}$$

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Gravity action	MacDowell-Mansouri action

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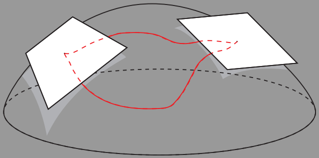
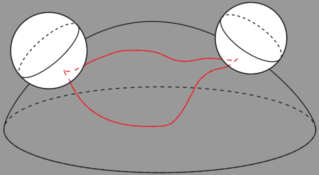
The key to the this approach is that the Lie algebra has a Killing-orthogonal splitting not as Lie algebras but as vector spaces, and is invariant under $SO(3,1)$:

$$\mathfrak{so}(4,1) \cong \mathfrak{so}(3,1) \oplus \mathbb{R}^{3,1}$$

- For understanding geometric meaning of splitting the easiest way is to first consider a lower-dimensional example, involving $SO(3)$ and $SO(2)$.
- An oriented 2d Riemannian manifold is often thought of in terms of an $SO(2)$ connection since, in the tangent bundle, parallel transport along two different paths from x to y gives results differing by a rotation of the tangent vector space at y.



- In this context, we can ask the geometric meaning of extending the gauge group from $SO(2)$ to $SO(3)$. The group $SO(3)$ acts naturally not on the bundle TM of tangent *vector spaces*, but on some bundle SM of 'tangent *spheres*'.
- Since $SO(3)$ acts to rotate the sphere, an $SO(3)$ connection on a 2d Riemannian manifold may be viewed as a rule for 'parallel transport' of tangent spheres.



An obvious way to get such an $SO(3)$ connection is simply to **roll a ball on the surface**, without twisting or slipping.

- Results of rolling a ball along two paths from x to y will differ by an element of $SO(3)$. Such group elements encode geometric information about the surface itself.

In our example, just as in the extension from the Lorentz group to the de Sitter group, we have an orthogonal splitting of the Lie algebra

$$\mathfrak{so}(3) \cong \mathfrak{so}(2) \oplus \mathbb{R}^2$$

given in terms of matrix components by:

$$\begin{bmatrix} 0 & u & a \\ -u & 0 & b \\ -a & -b & 0 \end{bmatrix} =$$

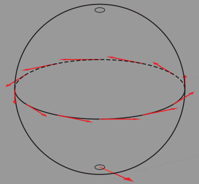
$$= \begin{bmatrix} 0 & u & \\ -u & 0 & \\ & & 0 \end{bmatrix} + \begin{bmatrix} & & a \\ & & b \\ -a & -b & \end{bmatrix}.$$

Like in the M-M case, this allows an $SO(3)$ connection A on an oriented 2d manifold to be split up into an $SO(2)$ connection ω and tetrad field e.

Geometric interpretation of these components: an infinitesimal rotation of the tangent sphere, as it begins to move along some path, breaks up into a parts:

- $\mathfrak{so}(2)$ part which gives an infinitesimal rotation around the axis through the point of tangency.

- $\mathbb{R}^2$ part which gives an infinitesimal translation of the point of tangency.



The connection thus defines a method of rolling a sphere along a surface.

Extrapolating from this example to the extension $SO(3,1) \subset SO(4,1)$, we sumarise a geometric interpretation for M-M gravity: the $SO(4,1)$ connection $A = (\omega, e)$ encodes the geometry of spacetime M by "rolling de Sitter spacetime along M" !

The *BF* reformulation

More recently, a different action for MacDowell-Mansouri gravity was proposed (by Freidel, Starodubtsev and Smolin) in which the MacDowell-Mansouri connection A is supplemented by an independent locally $so(4,1)$-valued 2-form B:

$$S = \int tr \left(B \wedge F - \frac{G\Lambda}{6} \widehat{B} \wedge \star \widehat{B} \right) .$$

This action is equivalent to the original MacDowell-Mansouri action by substituting the algebraic field equation which comes from variation over B back into the action.

However, written in this new form, Macdowell-Mansouri gravity has the appearance of a "deformation" of a topological gauge theory, so called the "*BF* theory".

The symmetry breaking occurs here only in the second term, with a dimensionless coefficient $G\Lambda \sim 10^{-120}$, suggesting that general relativity is in some sense "not too far" from a topological field theory.

- New proposed form of the action for gravity: $S = \int (B \wedge F - \beta B \wedge B - \frac{\alpha}{2} B \wedge * B)$

$$S = \int_{\mathcal{M}} \left(B_{\mu\nu IJ} F^{IJ}_{\rho\sigma} - \frac{\beta}{2} B_{\mu\nu IJ} B^{IJ}_{\rho\sigma} - \frac{\alpha}{4} \epsilon_{ijkl} B^{ij}_{\mu\nu} B^{kl}_{\rho\sigma} \right) \epsilon^{\mu\nu\rho\sigma} d^4 x$$

- where parameters $\alpha,\ \beta,\ \ell$ are connected to physical ones through:

$$\gamma = \frac{\beta}{\alpha}, \qquad \frac{1}{\ell^2} = \frac{\Lambda}{3}, \qquad \frac{\alpha}{2(\alpha^2 + \beta^2)\ell^2} = \frac{1}{2G}$$

- As the result we have standard action for gravity with cosmological constant Λ and dimensionless Immirzi parameter γ:

$$S = \frac{1}{2G} \int \left((R^{ij} \wedge e^k \wedge e^l - \frac{\Lambda}{6} e^i \wedge e^j \wedge e^k \wedge e^l) \epsilon_{ijkl} - \frac{2}{\gamma} R^{ij} \wedge e_i \wedge e_j \right)$$

- plus topological invariants

$$+ \frac{(1+\gamma^2)}{G\gamma} \int (T_i \wedge T^i - R_{ij} \wedge e^i \wedge e^j) + \frac{3\gamma}{2G\Lambda} \int R_{ij} \wedge R^{ij} - \frac{3}{4G\Lambda} \int R_{ij} \wedge R_{kl} \epsilon^{ijkl}$$

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What could we learn from this? Apparently three things:

- A. That

a D-dimensional Riemannian space can be conceived as one in which the usual SO(D) connection and the (appropriately normalized) vielbein are viewed as parts of an SO(D +1) connection. The torsion in this space is just a piece of the curvature two-form for the SO(D + 1) connection. So some of the components of the curvature tensor in the original spacetime could be interpreted as torsion in the reduced spaces.

- B. That

certain closed D-forms constructed out of the torsion two-form are topological invariants. With the construction A in mind, these invariants can be related to the Chern classess of SO(D) and SO(D + 1).

- C. That

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